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# THE FREE OSCILLATIONS OF FLUID ON A HEMISPHERE BOUNDED BY MERIDIANS OF LONGITUDE

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CONTENTS



A precise calculation is presented of the normal modes of oscillation of an ocean of uniform depth which is bounded by two meridians of longitude separated by an angle of 180°. The calculation takes full account of the horizontal divergence of the motion, and so is applicable to both barotropic and baroclinic modes of oscillation.

At small values of the parameter $\epsilon = 4\Omega^2R^2/gh$ (defined fully in §1) the calculation yields both the familiar gravity waves and also the nondivergent planetary waves computed in an earlier paper (Longuet-Higgins 1966). At large, positive values of $\epsilon$, corresponding to baroclinic waves, new types of oscillation appear in which the flux of energy is concentrated near the equator, the circuit being completed by Kelvin waves along the meridianal boundaries. The calculated frequencies are compared with asymptotic expressions derived from a recent $\beta$-plane analysis by D. W. Moore.

Solutions are also found corresponding to negative values of $\epsilon$. These must be included in a complete calculation of the response of the ocean to external forces. At small values of $\epsilon$ these solutions resemble the planetary waves. At large (negative) values of $\epsilon$ they represent almost-inertial motions concentrated near the poles, having a phase-velocity towards the east and an amplitude modulated so as to vanish at the boundaries.

The calculations are relevant to the real ocean in so far as they show the kinds of oscillation that might be expected in any ocean basin including any section of the equator (or including a pole). They also indicate the degree of accuracy to be expected in computing the frequencies of the normal modes by $\beta$-plane methods.

## 1. Introduction

The problem of determining the spectrum of free oscillations of a hemispherical ocean centred on the equator is of some interest in dynamical oceanography, when one wishes to investigate the possible types of oscillation that may occur in ocean basins comparable in extent to the Atlantic

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or the Pacific. Such oscillations are also of interest in connexion with the theory of long-period tides and the response of the ocean to large-scale, varying wind stresses.

The rotation of the Earth gives rise to some interesting and sometimes unexpected types of motion, to which there is no precise analogue in a nonrotating system. We may define a non-dimensional parameter

$$\epsilon = 4\Omega^2R^2/gh,$$

where $\Omega$ denotes the angular velocity of the Earth, $R$ the mean radius, $g$ the acceleration of gravity and $h$ the depth of water (assumed uniform and small compared to $R$). When this parameter is small (and only when it is small) two types of wave may be distinguished, namely the well-known *gravity* waves or waves of the first class, and the *planetary* waves or waves of the second class, which as $\epsilon$ tends to zero, reduce to steady currents (Margules 1893; Hough 1898). The gravity waves are, in the case of a hemispherical ocean, relatively simple to analyse; and as a preliminary to the present study, the spectrum of planetary waves in a hemispherical ocean has also been calculated (Longuet-Higgins 1966). The latter motions are the same as would take place in a thin rotating shell of uniform depth, if the horizontal divergence, and hence the vertical displacement of the surface, were assumed to be negligible.

In practice, however, the divergence cannot be entirely neglected, and one must assume at least a moderately large value of $\epsilon$ (of order 20). Moreover for internal, or baroclinic motions in the ocean the appropriate value of $\epsilon$ is two orders of magnitude greater than for barotropic motions. At the larger values of $\epsilon$ it is found that the two classes of motion mentioned above cannot be so easily distinguished, and in addition other types of motion are then possible (Longuet-Higgins 1968*a*). For example, one may have a type of wave travelling eastwards along the equator and similar in many respects to a Kelvin wave, confined by coriolis forces to the equatorial zone. Other motions confined to the polar regions also become possible.

In the study just mentioned (1968*a*) no meridianal boundaries were assumed; the fluid was imagined as covering the whole globe. Nevertheless, somewhat similar types of motion may be expected in the present problem, where the ocean is bounded by meridians of longitude. Thus it has been suggested (Longuet-Higgins 1968*a*, p. 576) that for large values of $\epsilon$ there is probably a solution for the hemisphere which represents a Kelvin wave propagated along the equator as far as the eastern boundary, where the energy divides and is propagated polewards along the eastern boundary, then equatorwards down the western boundary. A search for such a solution has since been carried out by Moore (1968) assuming an ocean with rectangular sides and using an equatorial $\beta$-plane approximation. Moore's analysis is valid at large values of $\epsilon$ and for rectangles whose meridianal extent is not more than a few degrees.

The present study, begun in 1966, had as its object the exact calculation of the normal modes in a rotating ocean bounded by meridians 0 and 180°, for all real values of the parameter $\epsilon$, and taking full account of the horizontal divergence. At small values of $\epsilon$ we find the gravity waves and the planetary waves as expected (§ 7). At large values of $\epsilon$ several other types of motion appear, one of which can be identified as the expected equatorial Kelvin wave (§ 9). An analogous anti-Kelvin wave, which is antisymmetric about the equator, is also found. However, because of the meridianal boundaries and the consequent non-separability of the longitudinal dependence, such simple modes are found only in certain well-defined ranges of the frequency, for any given value of $\epsilon$. Outside these ranges a statistical approach to the spectrum may be more appropriate (see § 10).

Besides the solutions for positive $\epsilon$, some solutions are also found for $\epsilon$ negative; these are discussed in § 11; they are relevant to the problem of forced oscillations. At large negative values of $\epsilon$ the energy in these motions tends to be concentrated near the two poles, as on the unbounded sphere (Longuet-Higgins 1968 *a*).

Over certain ranges of the parameter $\epsilon$, the calculated frequencies are found to agree well with approximate formulae derived from equatorial or polar $\beta$-plane theory (see §§ 9 and 11). This suggests that the frequency spectrum of the hemispherical ocean basin is representative of any ocean basin bounded by meridians of longitude. In certain respects the spectrum may be typical of any ocean enclosing a section of the equator.

## 2. Method of calculation

The method suggested by Goldsbrough (1933) for determining the free oscillations in ocean basins bounded by meridians was first tried, and was found to lead to series for the potential and streamfunction which were nonuniformly convergent in the neighbourhood of the boundaries.† We therefore resorted to a much more general method, valid for ocean basins of arbitrary shape and for any reasonable law of depth, which was first developed by Proudman (1916).

Let $\theta$ and $\phi$ denote the colatitude and the longitude (in radian measure) so that $\theta = 0$, $\pi$ correspond to the north and south poles respectively, while $\phi = 0$, $\pi$ define the meridianal boundaries. Let $\xi$ and $\eta$ denote the displacement of a particle from its equilibrium position in the directions of $\theta$ and $\phi$ increasing, and let $\zeta$ denote the displacement in the vertical direction. Then Proudman's main result, specialized to the case of the hemispherical basin, and when the radius $R$ of the sphere is taken to be unity, is that there exist in general two functions $\Phi$ and $\Psi$ (analogous to potential and streamfunction) such that everywhere in the interior of the basin

$$\left.\begin{aligned} \xi &= -\frac{\partial \Phi}{\partial \theta} - \frac{1}{h \sin\theta}\frac{\partial \Psi}{\partial \phi}, \\ \eta &= -\frac{1}{\sin\theta}\frac{\partial \Phi}{\partial \phi} + \frac{1}{h}\frac{\partial \Psi}{\partial \theta}, \end{aligned}\right\} \tag{2.1}$$

while at the boundaries $\phi = 0, \pi$,

$$h\frac{\partial \Phi}{\partial \phi} \to 0, \qquad \Psi/h \to 0. \tag{2.2}$$

It will be noted that the last conditions imply more than simply $\eta = 0$ at the boundaries. Proudman shows moreover that $\Phi$, $\Psi$ and $\zeta$ may be expanded in the forms

$$\left.\begin{aligned} \Phi &= \sum_{r=1}^{\infty} p_r \Phi_r, \\ \Psi &= \sum_{r=1}^{\infty} p_{-r} \Psi_r, \\ \zeta &= -\sum_{r=1}^{\infty} p_r \Phi_r, \end{aligned}\right\} \tag{2.3}$$

in which the functions $\Phi_r$ satisfy

$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(h\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right]\Phi_r + \mu_r \Phi_r = 0 \tag{2.4}$$

† Another disadvantage of this method is the asymmetry of the matrices involved.

in the interior, and
$$h\,\partial\Phi_r/\partial\phi \to 0 \tag{2.5}$$
at the boundaries $\phi = 0, \pi$, the $\mu_r$ being constant eigenvalues. Similarly,
$$\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{h}\frac{\partial}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{h\sin\theta}\frac{\partial}{\partial\phi}\right)\right]\Psi_r+\nu_r\Psi_r = 0 \tag{2.6}$$
with
$$\Psi_r/h \to 0 \tag{2.7}$$
at the boundaries, the $\nu_r$ being eigenvalues. Then the coefficients $p_r$, $p_{-r}$ in equations (2.3) are determined by the simultaneous equations
$$\left.\begin{aligned}\frac{d^2p_r}{dt^2}+2\Omega\sum_{s=-\infty}^{\infty}\beta_{r,s}\frac{dp_s}{dt}+g\mu_r p_r &= 0,\\ \frac{d^2p_{-r}}{dt^2}+2\Omega\sum_{s=-\infty}^{\infty}\beta_{-r,s}\frac{dp_s}{dt} &= 0,\end{aligned}\right\} \tag{2.8}$$
the suffix $r$ running through positive integer values. In equations (2.8) the coefficients $\beta_{r,s}$ are defined by
$$\left.\begin{aligned}\beta_{r,s} &= -\iint\frac{h\cos\theta}{\sin\theta}\left(\frac{\partial\Phi_r}{\partial\theta}\frac{\partial\Phi_s}{\partial\phi}-\frac{\partial\Phi_r}{\partial\phi}\frac{\partial\Phi_s}{\partial\theta}\right)dS,\\ \beta_{-r,s} &= -\iint\cos\theta\left(\frac{\partial\Psi_r}{\partial\theta}\frac{\partial\Phi_s}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial\Psi_r}{\partial\phi}\frac{\partial\Phi_s}{\partial\phi}\right)dS,\\ \beta_{r,-s} &= \iint\cos\theta\left(\frac{\partial\Phi_r}{\partial\theta}\frac{\partial\Psi_s}{\partial\theta}+\frac{1}{\sin^2\theta}\frac{\partial\Phi_r}{\partial\phi}\frac{\partial\Psi_s}{\partial\phi}\right)dS,\\ \beta_{-r,-s} &= -\iint\frac{\cos\theta}{h\sin\theta}\left(\frac{\partial\Psi_r}{\partial\theta}\frac{\partial\Psi_s}{\partial\phi}-\frac{\partial\Psi_r}{\partial\phi}\frac{\partial\Psi_s}{\partial\theta}\right)dS,\end{aligned}\right\} \tag{2.9}$$
the integrals being taken over the area of the basin and $dS$ denoting $\sin\theta\,d\theta\,d\phi$. These coefficients have been called ‘gyroscopic’ coefficients (Proudman 1916).

## 3. Application to the hemisphere

It can be seen immediately that when the depth $h$ is independent of $\theta$ and $\phi$ the solutions of equations (2.4) and (2.5) appropriate to the hemisphere are the spherical harmonics
$$\Phi_r = C_n^m P_n^m(\cos\theta)\cos m\phi, \tag{3.1}$$
where
$$\left.\begin{aligned}m &= 0, 1, 2, \ldots,\\ n &= m, (m+1), (m+2), \ldots,\end{aligned}\right\} \tag{3.2}$$
and
$$\mu_r = n(n+1)\,h. \tag{3.3}$$
The $P_n^m$ are the associated Legendre polynomials defined by
$$P_n^m(z) \equiv \frac{(1-z^2)^{\frac{1}{2}m}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}(z^2-1)^n, \tag{3.4}$$
and the $C_n^m$ are normalizing constants which we may choose so as to make
$$\iint h\left[\left(\frac{\partial\Phi_r}{\partial\theta}\right)^2+\left(\frac{1}{\sin\theta}\frac{\partial\Phi_r}{\partial\phi}\right)^2\right]dS = 1. \tag{3.5}$$

This implies $$C_n^m = \alpha_{n,m}/h^{\frac{1}{2}}, \tag{3.6}$$

where $$\alpha_{n,m} = \begin{cases} \left[\dfrac{1}{\pi}\dfrac{2n+1}{n(n+1)}\dfrac{(n-m)!}{(n+m)!}\right]^{\frac{1}{2}} & (m > 0), \\ \dfrac{1}{\pi}\dfrac{n+\frac{1}{2}}{n(n+1)} & (m = 0), \end{cases} \tag{3.7}$$

the case $n = m = 0$ being excluded.†

Similarly, the solutions of equations (2.6) and (2.7) appropriate to the present problem are given by $$\Psi_r = D_n^m P_n^m(\cos\theta)\sin m\phi, \tag{3.8}$$

where $$\left.\begin{aligned} m &= 1, 2, 3, \ldots, \\ n &= m, (m+1), (m+2), \ldots \end{aligned}\right\} \tag{3.9}$$

($m = 0$ is excluded) and where $$\nu_r = n(n+1)/h. \tag{3.10}$$

In order to make $$\iint \frac{1}{h}\left[\left(\frac{\partial\Psi_r}{\partial\theta}\right)^2 + \left(\frac{1}{\sin\theta}\frac{\partial\Psi_r}{\partial\phi}\right)^2\right]\mathrm{d}S = 1, \tag{3.11}$$

we choose $$D_n^m = h^{\frac{1}{2}}\alpha_{n,m}, \tag{3.12}$$

where $\alpha_{n,m}$ is given by (3.7).

It will be noticed that with each suffix $r$ is associated a pair of suffixes $\binom{m}{n}$, but this causes no difficulties, since the two-dimensional sequence $\binom{m}{n}$ may be ordered in a well-defined way (see § 4).

The coefficients $\beta_{r,s}$ defined by equations (2.9) can now be calculated. Details of the calculation, which is not quite straightforward, are given in the appendix. The results are as follows. If $\binom{m}{n}$ is the pair corresponding to the suffix $r$ and $\binom{m'}{n'}$ is the pair corresponding to the suffix $s$, then we have $$\beta_{\pm r, \pm s} = 0 \quad \text{when } (m+m') \text{ is even}, \tag{3.13}$$

and, when $(m+m')$ is odd,

$$\frac{\beta_{r,s}}{\alpha_r\alpha_s} = \left[\frac{n'(n'+1)+m'(m'+1)}{m'+1} - 2m'\frac{n(n+1)-n'(n'+1)+m'}{m^2+m'^2}\right] I\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} + \frac{1}{m'+1} I\begin{pmatrix} m & m'+2 \\ n & n' \end{pmatrix}, \tag{3.14}$$

$$\frac{\beta_{r,-s}}{\alpha_r\alpha_s} = \frac{2m}{(m'^2-m^2)(2n'+1)}\left[(n'-1)(n'+1)(n'+m')\, I\begin{pmatrix} m & m' \\ n & n'-1 \end{pmatrix} + n'(n'+2)(n'-m'+1)\, I\begin{pmatrix} m & m' \\ n & n'+1 \end{pmatrix}\right], \tag{3.15}$$

$$\beta_{-r,s} = -\beta_{s,-r}, \tag{3.16}$$

$$\frac{\beta_{-r,-s}}{\alpha_r\alpha_s} = \frac{2mm'}{m'^2-m^2} I\begin{pmatrix} m & m' \\ n & n' \end{pmatrix}, \tag{3.17}$$

where $$I\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} = \int_{-1}^{1} P_n^m(\mu)\, P_{n'}^{m'}(\mu)\, \mathrm{d}\mu. \tag{3.18}$$

† The inclusion of the constant term $C_0^0 P_0^0(\cos\theta)$ adds nothing to the particle displacements (2.1), which depend upon the derivatives of $\Phi$.

The integrals (3.18) have been studied and used in a previous paper (Longuet-Higgins 1966). From the fact that $P_n^m$ is an odd or even function of $\mu$ according as $(n-m)$ is odd or even, it is easy to see that (3.18) must vanish unless $[(n-m)+(n'-m')]$ is even. It follows that

$$\left.\begin{aligned} \beta_{r,s} &= 0 \quad \text{if} \quad [(n-m)+(n'-m')] \text{ is odd,} \\ \beta_{-r,s} &= 0 \quad \text{if} \quad [(n-m)+(n'-m')] \text{ is even,} \\ \beta_{r,-s} &= 0 \quad \text{if} \quad [(n-m)+(n'-m')] \text{ is even,} \\ \beta_{-r,-s} &= 0 \quad \text{if} \quad [(n-m)+(n'-m')] \text{ is odd.} \end{aligned}\right\} \tag{3.19}$$

The conditions (3.13) and (3.19) together imply that the solutions to equations (2.8) fall into two distinct sets as follows: in the first set

$$\left.\begin{aligned} r &= \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{even} \end{pmatrix}; \quad & s &= \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{odd} \end{pmatrix}, \\ -r &= \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix}; \quad & -s &= \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix} \end{aligned}\right\} \tag{3.20}$$

(or the same with $m$, $n$ interchanged with $m'$, $n'$). In this set of solutions $\Phi$ is symmetric about the equator (and so is $\zeta$) and $\Psi$ is antisymmetric. Alternatively

$$\left.\begin{aligned} r &= \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix}; \quad & s &= \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}, \\ -r &= \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{even} \end{pmatrix}; \quad & -s &= \begin{pmatrix} m' \\ n' \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{odd} \end{pmatrix} \end{aligned}\right\} \tag{3.21}$$

(or the same with $m$, $n$ interchanged with $m'$, $n'$). In this set $\Phi$ and $\zeta$ are antisymmetric about the equation and $\Psi$ is symmetric.

## 4. The free modes of oscillation

Let us seek solutions which vary harmonically with the time $t$. Thus in equations (2.8) let us write

$$p_r \propto \mathrm{e}^{-\mathrm{i}\sigma t}, \tag{4.1}$$

where $\sigma$ denotes the radian frequency. Then we have

$$\left.\begin{aligned} -\sigma^2 p_r - 2\mathrm{i}\sigma\Omega \sum_{s=-\infty}^{\infty} \beta_{r,s} p_s + n(n+1)\, gh p_r &= 0, \\ -\sigma^2 p_{-r} - 2\mathrm{i}\sigma\Omega \sum_{s=-\infty}^{\infty} \beta_{-r,s} p_s &= 0. \end{aligned}\right\} \tag{4.2}$$

Introducing the non-dimensional frequency

$$\lambda = \sigma/2\Omega, \tag{4.3}$$

we obtain, when $\lambda \neq 0$,

$$\left.\begin{aligned} [\lambda - n(n+1)\,\eta]\, p_r + \sum_{s=-\infty}^{\infty} \mathrm{i}\beta_{r,s} p_s &= 0, \\ \lambda p_{-r} + \sum_{s=-\infty}^{\infty} \mathrm{i}\beta_{-r,s} p_s &= 0, \end{aligned}\right\} \tag{4.4}$$

where

$$\eta = \frac{1}{\epsilon\lambda}, \qquad \epsilon = \frac{4\Omega^2 R^2}{gh}. \tag{4.5}$$

For any given value of $\eta$ in the range $(-\infty, \infty)$ we may now try to determine those values of $\lambda$ which allow a solution to the homogeneous system of equations (4.4); that is to say we seek the eigenvalues $\lambda_i$ of the matrix coefficients. Having determined the $\lambda_i$ the corresponding values of $\epsilon$ are found from the relation

$$\epsilon_i = 1/\eta\lambda_i \tag{4.6}$$

(cf. Longuet-Higgins 1968*a*, § 5).

Consider first the solutions with $\Phi$ symmetric and $\Psi$ antisymmetric. We may write

$$p_r = \begin{Bmatrix} A_r & \text{if} & \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{even} \end{pmatrix}, \\ \mathrm{i}B_r & \text{if} & \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{odd} \end{pmatrix}, \end{Bmatrix} \tag{4.7}$$

and

$$p_{-r} = \begin{Bmatrix} A_{-r} & \text{if} & \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{even} \\ \text{odd} \end{pmatrix}, \\ \mathrm{i}B_{-r} & \text{if} & \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} \text{odd} \\ \text{even} \end{pmatrix}, \end{Bmatrix} \tag{4.8}$$

and then (4.4) reduces to a system of *real* simultaneous equations for $(A_r, B_r, A_{-r}, B_{-r})$. Moreover, since $\beta_{r,s}$ is antisymmetric in $(r, s)$ the coefficients in the real system are symmetric in $(r, s)$. Hence the eigenvalues $\lambda_i$ are all real.

The ordering of the coefficients (when $\Phi$ is symmetric) may be carried out according to the following scheme:

$$\left.\begin{array}{llllllll} & & A_1 & A_2 & A_3 & A_4 & A_5 & \\ \begin{pmatrix} m \\ n \end{pmatrix} = & & \begin{pmatrix} 0 \\ 2 \end{pmatrix}, & \begin{pmatrix} 2 \\ 2 \end{pmatrix}; & \begin{pmatrix} 0 \\ 4 \end{pmatrix}, & \begin{pmatrix} 2 \\ 4 \end{pmatrix}, & \begin{pmatrix} 4 \\ 4 \end{pmatrix}; & \ldots \end{array}\right\}$$

$$\left.\begin{array}{llllllll} & B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & \\ \begin{pmatrix} m \\ n \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \end{pmatrix}; & \begin{pmatrix} 1 \\ 3 \end{pmatrix}, & \begin{pmatrix} 3 \\ 3 \end{pmatrix}; & \begin{pmatrix} 1 \\ 5 \end{pmatrix}, & \begin{pmatrix} 3 \\ 5 \end{pmatrix}, & \begin{pmatrix} 5 \\ 5 \end{pmatrix}; & \ldots \end{array}\right\}$$

$$\left.\begin{array}{llllllll} & A_{-1} & A_{-2} & A_{-3} & A_{-4} & A_{-5} & A_{-6} & \\ \begin{pmatrix} m \\ n \end{pmatrix} = & \begin{pmatrix} 2 \\ 3 \end{pmatrix}; & \begin{pmatrix} 2 \\ 5 \end{pmatrix}, & \begin{pmatrix} 4 \\ 5 \end{pmatrix}; & \begin{pmatrix} 2 \\ 7 \end{pmatrix}, & \begin{pmatrix} 4 \\ 7 \end{pmatrix}, & \begin{pmatrix} 6 \\ 7 \end{pmatrix}; & \ldots \end{array}\right\}$$

$$\left.\begin{array}{llllllll} & B_{-1} & B_{-2} & B_{-3} & B_{-4} & B_{-5} & B_{-6} & \\ \begin{pmatrix} m \\ n \end{pmatrix} = & \begin{pmatrix} 1 \\ 2 \end{pmatrix}; & \begin{pmatrix} 1 \\ 4 \end{pmatrix}, & \begin{pmatrix} 3 \\ 4 \end{pmatrix}; & \begin{pmatrix} 1 \\ 6 \end{pmatrix}, & \begin{pmatrix} 3 \\ 6 \end{pmatrix}, & \begin{pmatrix} 5 \\ 6 \end{pmatrix}; & \ldots \end{array}\right\}$$

It will be noticed that the term $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is a constant, has been omitted. Also in the sequence $A_{-r}$ there are no terms with $m = 0$.

In practical calculations we need to place an upper limit on the number of rows and columns in the matrix of equations (4.4). This may be done by including only those terms corresponding to spherical harmonics whose degree $n$ or $n'$ does not exceed a certain maximum value, say $N$. For example if $N = 6$ we should include $A_1$ to $A_9$, $B_1$ to $B_6$, $A_{-1}$ to $A_{-3}$ and $B_{-1}$ to $B_{-6}$, altogether 24 rows and columns. In general the number of rows and columns is equal to

$$\tfrac{1}{2}(N+1)^2 - 1 \quad \text{or} \quad \tfrac{1}{2}N(N+2)$$

according as $N$ is odd or even.


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Similarly, for the solutions with $\Phi$ antisymmetric and $\Psi$ symmetric about the equator we may adopt the following order for the coefficients:

$$\left.\begin{aligned}
\binom{m}{n} &= \overset{A_1}{\binom{0}{1}}; \quad \overset{A_2}{\binom{0}{3}}, \ \overset{A_3}{\binom{2}{3}}; \quad \overset{A_4}{\binom{0}{5}}, \ \overset{A_5}{\binom{2}{5}}, \ \overset{A_6}{\binom{4}{5}}; \ \ldots \\
\binom{m}{n} &= \overset{B_1}{\binom{1}{2}}; \quad \overset{B_2}{\binom{1}{4}}, \ \overset{B_3}{\binom{3}{4}}; \quad \overset{B_4}{\binom{1}{6}}, \ \overset{B_5}{\binom{3}{6}}, \ \overset{B_6}{\binom{5}{6}}; \ \ldots \\
\binom{m}{n} &= \overset{A_{-1}}{\binom{2}{2}}; \quad \overset{A_{-2}}{\binom{2}{4}}, \ \overset{A_{-3}}{\binom{4}{4}}; \quad \overset{A_{-4}}{\binom{2}{6}}, \ \overset{A_{-5}}{\binom{4}{6}}, \ \overset{A_{-6}}{\binom{6}{6}}; \ \ldots \\
\binom{m}{n} &= \overset{B_{-1}}{\binom{1}{1}}; \quad \overset{B_{-2}}{\binom{1}{3}}, \ \overset{B_{-3}}{\binom{3}{3}}; \quad \overset{B_{-4}}{\binom{1}{5}}, \ \overset{B_{-5}}{\binom{3}{5}}, \ \overset{B_{-6}}{\binom{5}{5}}; \ \ldots
\end{aligned}\right\}$$

(Now there is no need to omit the first element.) If $N$ denotes the maximum value of $n$ or $n'$, the total number of rows and columns included is now equal to

$$\tfrac{1}{2}(N+1)^2 \quad \text{or} \quad \tfrac{1}{2}N(N+2)$$

according as $N$ is odd or even.

## 5. Computation of the free modes

Because of the complication inherent both in the analysis and in the programming of the above solution for a digital computer, the authors made two quite independent preliminary calculations. One of these was programmed on an IBM 7094 in London and the other on the CDC 3600 at the University of California, San Diego. By comparison of the two a number of significant errors was discovered and eliminated.

The resulting eigenfrequencies were also checked in other ways. For example, as $\epsilon \to 0$ the eigenfrequencies tended to the values found previously in the case $\epsilon = 0$ (Longuet-Higgins 1966). And when $\eta$ was replaced by $-\eta$ then the eigenvalues were found to be multiplied by $-1$ (which would not be true of an arbitrary symmetric matrix whose diagonal terms were reversed).

For greater accuracy the program designed for the CDC 3600 was transferred, with the very slight modifications necessary to the CDC 6600 at the National Centre for Atmospheric Research at Boulder, Colorado. There it was found possible to carry the computation as far as $N = 19$ and to compare the eigenvalues with those for the previous approximation $N = 18$.†

Corresponding eigenfrequencies in the two approximations were then compared, and if these differed by less than one part in $10^{-3}$ they were plotted on a graph of $\lambda$ against $\epsilon$. The results for $\Phi$ symmetric are shown in figures 1 and 9, and those for $\Phi$ antisymmetric are shown in figures 2 and 10. Figures 1 and 2, for which $\epsilon > 0$, correspond to free oscillations on a physically realizable sphere with depth $h > 0$. These will be discussed first. The modes corresponding to negative values of (figures 9 and 10) will be discussed in § 12.

† For $\Phi$ symmetric. The computations for $\Phi$ antisymmetric are correct only as far as $N = 16$.

## 6. The solutions with $\epsilon$ positive

In many respects the curves of figures 1 and 2 resemble the corresponding curves for the normal modes over a complete sphere (Longuet-Higgins 1968*a*, figures 1 to 6). Thus on the right-hand side of each diagram (*small* positive values of $\epsilon$) the eigenfrequencies $\lambda$ are divided into two

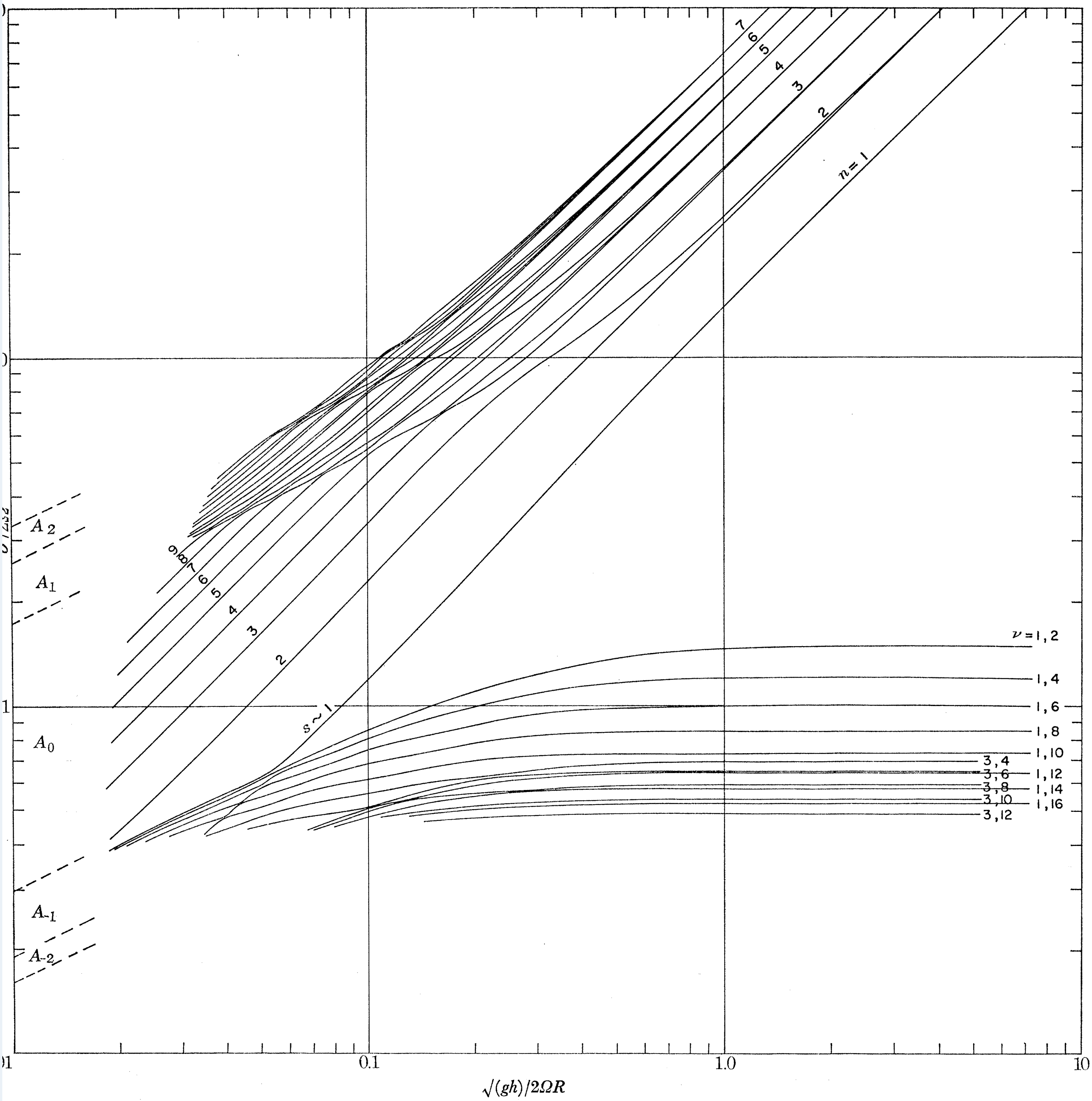


Figure 1. The computed eigenfrequencies $\lambda = \sigma/2\Omega$ of the oscillations of a hemispherical ocean basin of depth $h$, as a function of $\epsilon^{-\frac{1}{2}} = \sqrt{(gh)}/2\Omega R$, when $\Phi$ is symmetric about the equator.

distinct families tending either to values proportional to $\epsilon^{-\frac{1}{2}}$ (corresponding to gravity waves). On the left of the diagram, the frequencies tend to zero either like $\epsilon^{-\frac{1}{2}}$ or like $\epsilon^{-\frac{1}{4}}$, and moreover at intermediate frequencies some of the modes appear to cross from the upper family to the lower family or vice versa.

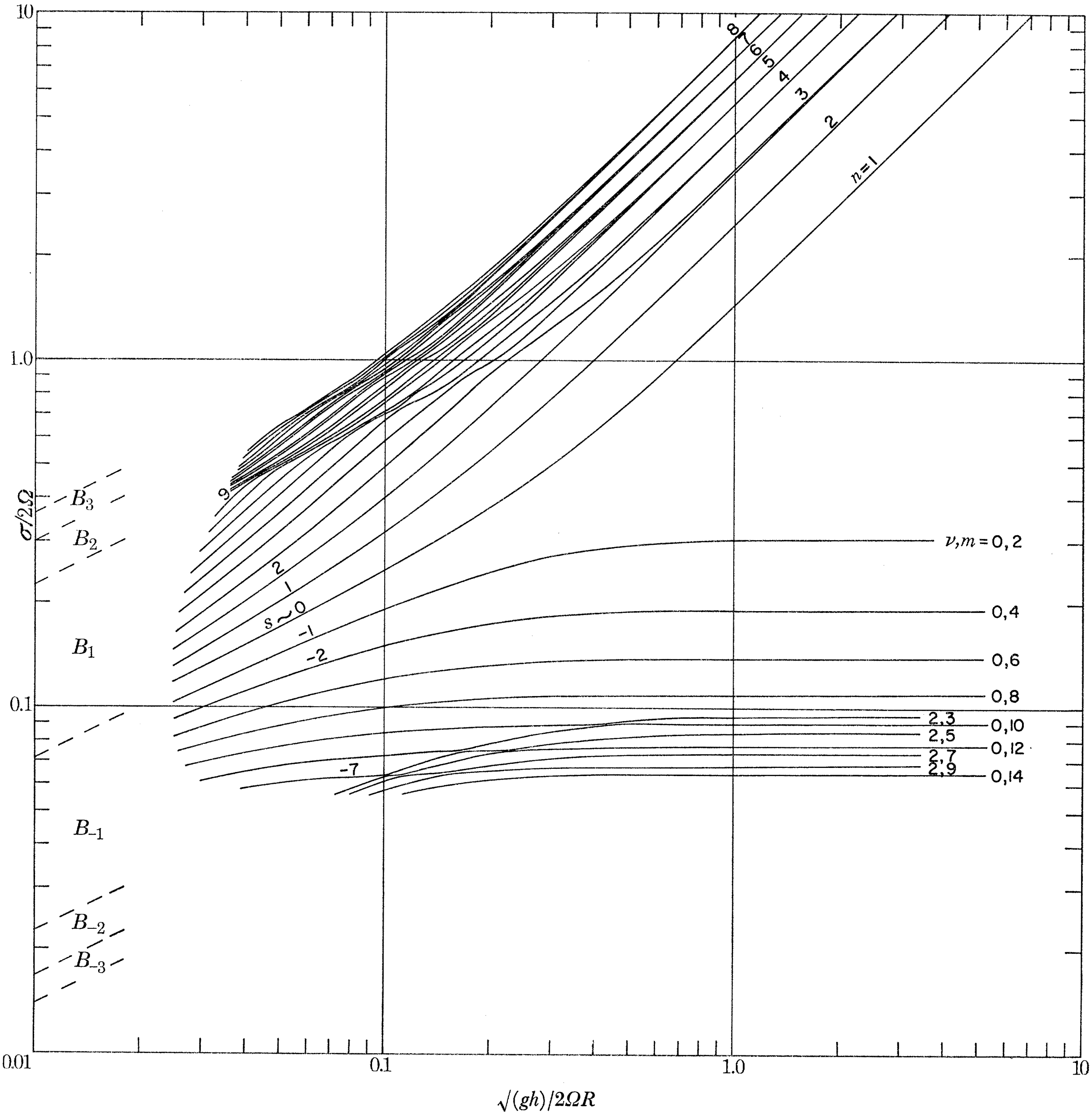


FIGURE 2. The computed eigenfrequencies $\lambda = \sigma/2\Omega$ of the oscillations of a hemispherical ocean basin of depth $h$ as a function of $\epsilon^{-\frac{1}{2}} = \sqrt{(gh)}/2\Omega R$, when $\Phi$ is antisymmetric about the equator.

In other respects the curves of figures 1 and 2 differ from those for the unbounded sphere. Thus (1) for the unbounded sphere it was possible to separate waves having a given latitudinal wavenumber $s$ (or $m$), and also to separate those waves progressing eastwards from those progressing westwards. In the presence of meridianal boundaries these separations can no longer be made. For, to satisfy the boundary conditions, modes having different values of $s$ are forced to combine. On the hemisphere, the only complete separation is between the symmetric and the antisymmetric modes (which are displayed together in figures 1 to 5 of Longuet-Higgins 1968 *a*).

(2) As $\epsilon \to 0$ (on the right of figure 1) the frequencies tend, not to the frequencies of the planetary waves on a complete sphere, but to those for a hemispherical ocean (calculated previously in Longuet-Higgins 1966).

(3) Because of the non-separability, many of the curves in figures 1 and 2 are very closely spaced in some regions of the diagram and appear almost to intersect. The interpretation of such regions will be discussed below.

First, however, we shall discuss the possible asymptotic forms of the solution as $\epsilon \to 0$ or $\epsilon \to \infty$ through positive values.

## 7. Asymptotic forms as $\epsilon \to 0$

When $\epsilon\lambda \ll 1$ then by (4.5) $\eta \gg 1$, and it is clear from (4.4) that the determinant of the system of equations (4.4) is approximated by

$$\prod_{r>0} [\lambda - n(n+1)\,\eta] \times \Delta(\lambda, \beta), \tag{7.1}$$

where $\Delta(\lambda,\beta)$ denotes the characteristic determinant of the elements $\beta_{-r,-s}$. The vanishing of the factors in the first product gives

$$\lambda = n(n+1)\,\eta = n(n+1)/\epsilon\lambda; \qquad \lambda = \sqrt{\{n(n+1)\}}\,\epsilon^{-\frac{1}{2}} \tag{7.2}$$

Table 1. Parameters for the asymptotic forms of the class I solutions (gravity waves) in the limit as $\epsilon \to 0$

| | | | $m$ | |
|---|---|---|---|---|
| $n$ | $n(n+1)$ | $\sqrt{\{n(n+1)\}}$ | $\Phi$ symmetric | $\Phi$ antisymmetric |
| 1 | 2 | 1.414 | 1 | 0 |
| 2 | 6 | 2.449 | 2, 0 | 1 |
| 3 | 12 | 3.464 | 3, 1 | 2,0 |
| 4 | 20 | 4.472 | 4, 2, 0 | 3, 1 |
| 5 | 30 | 5.477 | 5, 3, 1 | 4, 2, 0 |
| 6 | 42 | 6.481 | 6, 4, 2, 0 | 5, 3, 1 |
| 7 | 56 | 7.483 | 7, 5, 3, 1 | 6, 4, 2, 0 |

and corresponds to the solution

$$\left.\begin{aligned} \Phi &= C_n^m P_n^m(\cos\theta)\cos m\phi \quad (m \leqslant n), \\ \Psi &= 0. \end{aligned}\right\} \tag{7.3}$$

These are waves of Class I or standing gravity waves. We note that the frequency equation (7.2) may also be written as

$$\sigma^2 = n(n+1)\,gh. \tag{7.4}$$

However, corresponding to any one value of $n$ there may be more than one value of $m$. This accounts for the branching of the curves in the upper right-hand corners of figures 1 and 2. The first few possibilities are listed in table 1. Among the symmetric modes, for instance, it will be seen that there is only one mode corresponding to $n = 1$, two each corresponding to $n = 2$ and 3, three corresponding to $n = 4$ and 5, and so on. Also it may be remarked that since in equation (7.3)





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$\cos m\phi$ can be expressed as $\frac{1}{2}(e^{im\phi}+e^{-im\phi})$ each solution may be thought of as the sum of two gravity waves progressing round the pole in opposite directions.

The vanishing of $\Delta(\lambda, \beta)$, on the other hand, gives the waves of Class II, or the planetary waves. In each of these, the non-dimensional frequency $\lambda = \sigma/2\Omega$ tends to a finite value independent of $\Omega$ or $\epsilon$. It was shown in Longuet-Higgins (1966) that the lower modes have the form of sinusoidal oscillations with phase progressing towards the west. The *amplitude* of the stream-function, however, is modulated in such a way as always to vanish at the boundary. The limiting frequencies $\lambda$ of these waves are shown in table 2.

Table 2. Computed values of the frequencies of the class II solutions (planetary waves) in the limit as $\epsilon \to 0$

| $\Psi$ symmetric | | | | | $\Psi$ antisymmetric | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\bar{n}$ | $\bar{m}$ | $\nu$ | $\frac{\bar{m}}{\bar{n}(\bar{n}+1)}$ | $\lambda$ | $\bar{n}$ | $\bar{m}$ | $\nu$ | $\frac{\bar{m}}{\bar{n}(\bar{n}+1)}$ |
| .3077 (6) | 2 | 2 | 0 | .3330 | — | – | – | – | — |
| .1919 | 4 | 4 | 0 | .2000 | .1497 | {3<br>4} | 2<br>3 | 1<br>1 | .1667<br>.1500 |
| .1395 (4) | 6 | 6 | 0 | .1429 | .1214 | {5<br>6} | 4<br>5 | 1<br>1 | .1333<br>.1190 |
| .1096 | 8 | 8 | 0 | .1111 | .1002 | {7<br>8} | 6<br>7 | 1<br>1 | .1071<br>.0972 |
| .0957 | 5 | 3 | 2 | .1000 | .0847 | {9<br>10} | 8<br>9 | 1<br>1 | .0888<br>.0818 |
| .0907 (6) | 10 | 10 | 0 | .0909 | .0733 | {11<br>12} | 10<br>11 | 1<br>1 | .0758<br>.0705 |
| .0850 | 7 | 5 | 2 | .0893 | .0698 | 7 | 4 | 3 | .0714 |
| .0779 (8) | 12 | 12 | 0 | .0769 | .0651 | 9 | 6 | 3 | .0667 |
| .0741 (39) | 9 | 7 | 2 | .0777 | .0645 (4) | {13<br>14} | 12<br>13 | 1<br>1 | .0659<br>.0619 |
| .0684 (3) | 11 | 9 | 2 | .0682 | .0595 (3) | 11 | 8 | 3 | .0606 |
| .065 (3) | 14 | 14 | 0 | .067 | .057 (4) | {15<br>16} | 14<br>15 | 1<br>1 | .058<br>.055 |
| .061 (59) | 13 | 11 | 2 | .060 | .053 (2) | 13 | 10 | 3 | .055 |
| .056 (5) | 9 | 5 | 4 | .056 | .047 (—) | 15 | 12 | 3 | .050 |
| .055 (3) | 15 | 13 | 2 | .054 | .045 | 11 | 6 | 5 | .045 |
| .052 (1) | 11 | 7 | 4 | .053 | .044 (3) | 13 | 8 | 5 | .044 |

In the first column of table 2 are the non-dimensional frequencies for $\Psi$ symmetric, calculated when $N = 16$. The frequencies for $N = 15$ are shown in parentheses below each entry, when these differ at all from $N = 16$. In the next three columns of table 2 are the values of $n$, $m$ and $\nu = (n-m)$ corresponding to the dominant harmonic of each mode. These can generally be determined from an examination of the corresponding eigenvalues (see Longuet-Higgins 1966).

In the fifth column of the table are shown the values of $m/n(n+1)$, which would give the frequencies of the harmonic $(n, m)$ on the unbounded sphere. The first and fifth columns may be compared.

Similar parameters but for $\Psi$ antisymmetric are shown on the right of table 2. When $\nu = 1$ it is found that there are generally two dominant harmonics of about the same magnitude, so both of these are indicated.

It is clear that those entries dominated by harmonics of order $n \ll N$ are more reliably determined than those for which $n$ approaches $N$. Those for which $n$ exceeds $N$ cannot appear in the table. It will be seen also that those harmonics for which $m < 1\sqrt{\{\nu(\nu+1)\}}$ do not dominate any mode. This may be connected with the fact that if

$$\frac{m}{n(n+1)} = \frac{m}{(m+\nu)\,(m+\nu+1)} = \lambda_{m,\nu}, \tag{7.5}$$

say, then for a given value of $\nu$, $\lambda_{m,\nu}$ is a maximum when $m = \sqrt{\{\nu(\nu+1)\}}$. Hence harmonics with $m < \sqrt{\{\nu(\nu+1)\}}$ can generally be dominated by other harmonics with the same value of $\nu$ and about the same frequency, but with a larger value of $m$.

## 8. Asymptotic forms as $\epsilon \to \infty$ on an unbounded $\beta$-plane

In discussing the asymptotic forms of the solutions as $\epsilon \to \infty$ it will be helpful first to recall the simpler situation when there are no meridianal boundaries and the fluid covers the whole sphere. In that situation it has been shown (Longuet-Higgins 1968*a*) that as $\epsilon \to \infty$, so the energy of the relative motion tends to be concentrated in a narrow zone near the equator whose width is of order $\epsilon^{-\frac{1}{4}}$ (the radius of the sphere being unity). Consequently for large values of $\epsilon$ we may use the equatorial $\beta$-plane approximation (Rattray 1964; Matsuno 1966).

In this approximation, if $(x, y)$ denote coordinates taken eastwards and northwards respectively, and if $(u, v)$ denote the corresponding components of the velocity, the equations of motion and of continuity become simply

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - 2\Omega y v + \frac{\partial p}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + 2\Omega y u + \frac{\partial p}{\partial y} &= 0, \\ \frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + \frac{1}{gh}\frac{\partial p}{\partial t} &= 0. \end{aligned}\right\} \tag{8.1}$$

We are interested in harmonic solutions where $u$, $v$ and $p$ are proportional to $\exp\{\mathrm{i}(mx - \sigma t)\}$. Here $m$ is an east–west wavenumber which near the equator is equivalent to the upper order of the spherical harmonic. Thus in (8.1) we replace $\partial/\partial x$ and $\partial/\partial t$ by $\mathrm{i}m$ and $-\mathrm{i}\sigma$ respectively. By introducing also the scaled coordinates

$$\left.\begin{aligned} (\xi, \eta) &= \epsilon^{\frac{1}{4}}(x, y)/R, \qquad & \tau &= \epsilon^{-\frac{1}{4}} 2\Omega t, \\ K &= \epsilon^{-\frac{1}{4}} m R, \qquad & L &= \epsilon^{\frac{1}{4}}\lambda, \end{aligned}\right\} \tag{8.2}$$

with

we reduce equations (8.1) to the form

$$\left.\begin{aligned} -\mathrm{i}Lu' - \eta v' + \mathrm{i}Kp' &= 0, \\ \eta u' - \mathrm{i}Lv' + Dp' &= 0, \\ \mathrm{i}Ku' + Dv' - \mathrm{i}Lp' &= 0, \end{aligned}\right\} \tag{8.3}$$

where $(u',v') = (u,v)/\sqrt{(gh)}$, $p' = p/gh$ and $D$ denotes $\partial/\partial\eta$. Eliminating $u'$ and $p'$ from these equations we obtain for $v'$ the differential equation

$$D^2v' + [(L^2 - K^2 - K/L) - \eta^2]\,v' = 0, \tag{8.4}$$

with the conditions that as $\eta \to \pm\infty$ the solution must be bounded. Equation (8.4) is a form of Weber's equation. To satisfy the boundary conditions we must have

$$(L^2 - K^2 - K/L) = 2\nu + 1, \tag{8.5}$$

where $\nu$ is a positive integer or zero. The corresponding expressions for $u'$, $v'$ and $p'$ are given by

$$\left.\begin{aligned} \mathrm{i}v' &\propto (L^2-K^2)\,W_\nu,\\ u' &\propto \nu(L-K)\,W_{\nu-1} + \tfrac{1}{2}(L+K)\,W_{\nu+1},\\ p' &\propto \nu(L-K)\,W_{\nu-1} - \tfrac{1}{2}(L+K)\,W_{\nu+1},\end{aligned}\right\} \tag{8.6}$$

where $$W_\nu \equiv \exp\{-\tfrac{1}{2}\eta^2\}\,H_\nu(\eta)\,\exp\{\mathrm{i}(K\xi - L\tau)\} \tag{8.7}$$

and $H_\nu(\eta)$ denotes the Hermite polynomial of degree $\nu$. From (8.6) we see that $\nu$ signifies the number of zeros of the function $v$, that is to say the number of distinct latitudes for which the northwards component of velocity $v$ vanishes. If $\nu$ is odd the equator is always one such latitude, and in fact the pressure is symmetric about the equator. If $\nu$ is even the pressure is antisymmetric about the equator.

The dispersion relation (that is the relation between the non-dimensional frequency $L$ and the east–west wavenumber $K$), is given by equation (8.5), and may be written

$$L^3 - (K^2 + 2\nu + 1)\,L - K = 0. \tag{8.8}$$

Since this equation is cubic in $L$ and quadratic in $K$, it follows that for any given value of $K$ there are in general three possible values of $L$, and for a given value of $L$ there are two values of $K$ given by

$$K = -\frac{1}{2L} \pm \sqrt{\left\{\left(\frac{1}{2L} - L\right)^2 - 2\nu\right\}}. \tag{8.9}$$

These are real provided the expression under the square root is non-negative.

Two provisos must be noted: (1) in the special case $\nu = 0$ equation (8.8) may be factorized:

$$(L+K)\,(L^2 - LK - 1) = 0. \tag{8.10}$$

The quadratic roots are valid; but it turns out that the linear root does not provide a solution of the original equations and so must be disallowed.

On the other hand, there does exist a solution with

$$(L-K) = 0, \tag{8.11}$$

in which $v$ vanishes identically and $u$, $p$ are both proportional to $W_0$. This solution is shown in figure 3 where it is indicated formally by the notation $\nu = -1$.

To interpret these solutions we note first that because of the factor $\mathrm{e}^{-\frac{1}{2}\eta^2}$ in $W_\nu$ the motion is negligible when $|\eta|$ is much greater than 1, that is to say when $|y|$ is much greater than $\epsilon^{-\frac{1}{4}}$, so that for large $\epsilon$ the motion is indeed confined to the equatorial zone.

Secondly, we see from (8.6) that the pressure (or surface elevation) is always in phase with the east–west velocity and in quadrature with the north–south velocity.

Thirdly, because of the factor $\exp\{i(K\xi - L\eta)\}$ in (8.6) and (8.7) the motions are progressive towards the east or west according as the phase velocity

$$C = \frac{L}{K} = \epsilon^{\frac{1}{2}}\left(\frac{\lambda}{m}\right) = \frac{\sigma/m}{\sqrt{(gh)}} \tag{8.12}$$

is positive or negative.

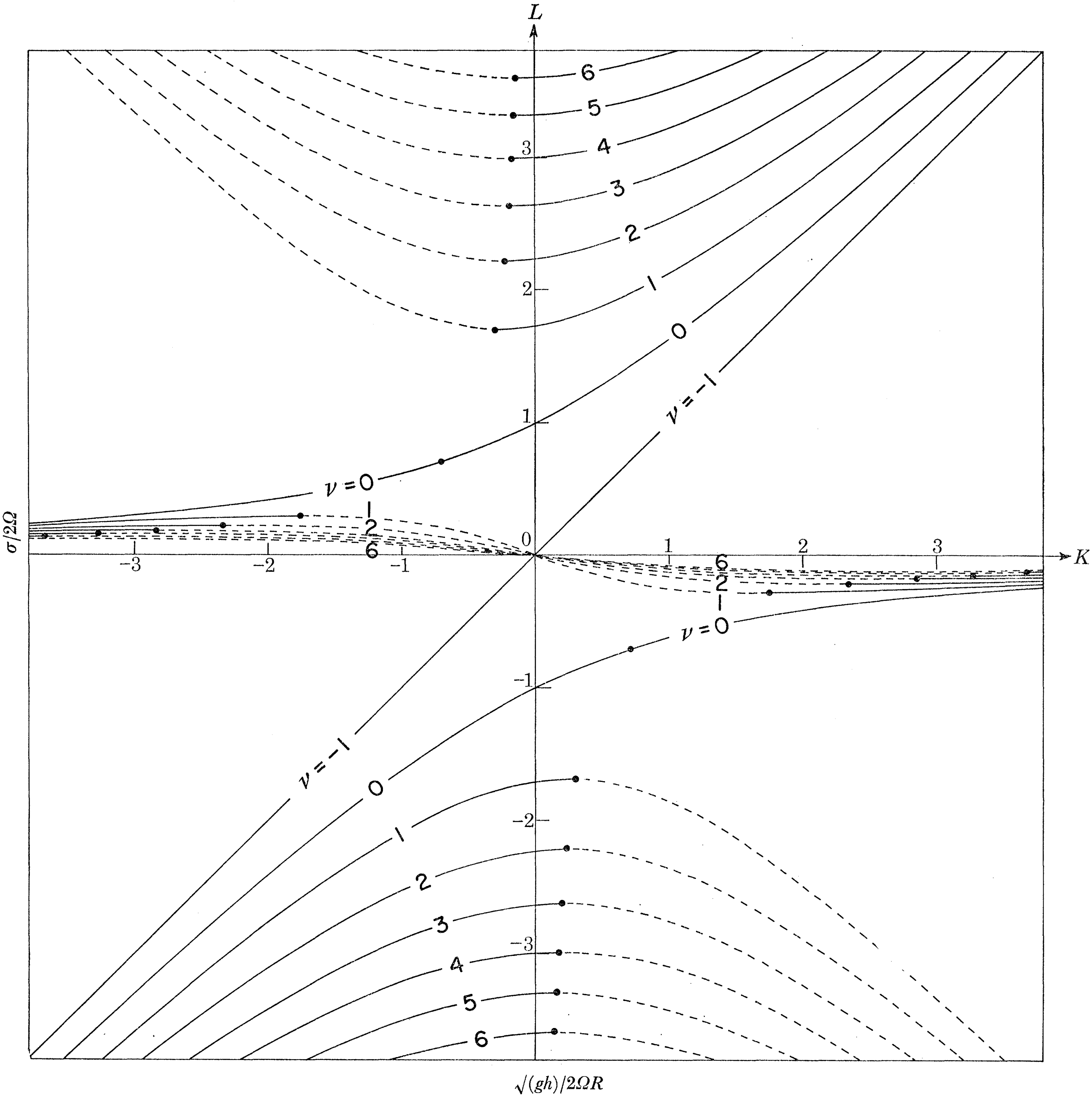


FIGURE 3. The dispersion relation between the scaled frequency $L = \epsilon^{\frac{1}{4}}(\sigma/2\Omega)$ and the scaled wavenumber $K = \epsilon^{-\frac{1}{4}}(mR)$ for the equatorial $\beta$-plane waves ($\epsilon \gg 1$).



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From figure 3 it appears that for general values of $\nu$ there are two types of waves:

(1) *Type* 1 *waves* in which $L^2 > 1$; these may progress either eastwards or westwards. For a given value of $\nu > 0$ there is a *minimum* frequency given by the vanishing of the radical in equation (8.9), hence

$$L^2 = (\nu + \tfrac{1}{2}) + \sqrt{\{\nu(\nu+1)\}}. \tag{8.13}$$

For each value of $L^2$ exceeding this value there are two waves either both travelling westwards or one travelling westwards and the other eastwards. The corresponding group-velocities (which are given by the gradients of the curves in figure 3) are always of opposite sign.

(2) *Type* 2 *waves* in which $L^2 < 1$; these waves always progress towards the west. For a given value of $\nu > 0$ there is a *maximum* frequency given by

$$L^2 = (\nu + \tfrac{1}{2}) - \sqrt{\{\nu(\nu+1)\}}. \tag{8.14}$$

For each value of $L^2$ less than this value there appear to be two waves, both travelling westwards but with group velocities of opposite sign.

The cases $\nu = 0$ and $\nu = -1$ require special consideration. For any given value of $K$ or $L$ there exists a wave of each kind. We may call these waves of type 3. The wave corresponding to $\nu = -1$ is symmetric about the equator. Its phase velocity is always towards the east and is given by

$$\sigma/m = \sqrt{(gh)}\,(L/K) = \sqrt{(gh)}. \tag{8.15}$$

It is in fact analogous to a Kelvin wave trapped at the equator (Longuet-Higgins 1968*a*). Its group-velocity is also equal to $\sqrt{(gh)}$, in the same direction as the phase velocity.

The wave corresponding to $\nu = 0$ is antisymmetric about the equator. Its group-velocity is always towards the east but it may have a phase velocity which is eastwards or westwards according as $L^2 \gtrless 1$. It may be called an anti-Kelvin wave.

When $L^2 \gg 1$ both these waves take on some of the characteristics of a wave of type 1, and when $L^2 \ll 1$ the anti-Kelvin wave ($\nu = 0$) takes on some of the characteristics of a wave of type 2.

It must be emphasized that the above approximations are valid only if $\epsilon^{-\frac{1}{4}} \ll 1$ that is to say if the latitudinal scale of the motion, which is proportional to $\epsilon^{-\frac{1}{4}}$, is a small fraction of a quadrant of the globe. This implies, for example, that if the east–west wavenumber $m$ were of order unity, then $K = \epsilon^{-\frac{1}{4}}m$, would be a small quantity; hence only those parts of figure 3 which lay close to the $L$-axis would be of any significance.

To translate figure 3 into the same terms as figures 1 and 2, we replot the curves with new axes:

$$K^2 = m^2\epsilon^{-\frac{1}{2}}, \quad |KL| = |m\lambda|, \tag{8.16}$$

and on a logarithmic scale. The symmetric modes are shown in figure 4 and the antisymmetric modes in figure 5. For any given wavenumber, say $m = 1$ the curves show the non-dimensional frequency $|\lambda|$ as a function of the parameter $\epsilon^{-\frac{1}{2}} = \sqrt{(gh)}/2\Omega$, just as in figures 1 and 2. For any other wavenumber $m$ the curves of $|\lambda|$ as a function of $\epsilon^{-\frac{1}{2}}$ are precisely similar in form, being displaced by a factor $m$ downwards and by a factor of $m^2$ to the left, that is to say they are displaced parallel to the tangent lines in figures 4 and 5, which have a gradient of 1 : 2. The approximation is valid so long as $\epsilon^{-\frac{1}{4}} \ll 1$, that is, so long as the abscissa in figures 4 and 5 is small compared to $m^2$. The appropriate values of $m$ are determined, as we shall see, by the boundary conditions.

The curves in figure 4 (the symmetric modes) correspond to odd values of $\nu$. Thus the straight line at 45°, corresponding to $\nu = -1$, is the Kelvin wave. The curves of type 1 (above the Kelvin wave) have slopes parallel to $|\lambda| = \epsilon^{-\frac{1}{2}}$ when $\epsilon^{-\frac{1}{2}}$ is large, and parallel to $|\lambda| = \epsilon^{-\frac{1}{4}}$ when $\epsilon^{-\frac{1}{2}}$ is small. Figure 4 contains both eastward-going waves ($KL > 0$) and westward-going waves ($KL < 0$). The Kelvin wave propagates eastwards, the type 1 waves part eastwards and part westwards. Also shown in figure 4 are the tangents to the frequency curves parallel to $L$ = constant (broken lines). As in figure 3, these tangents divide the corresponding curve where the radical in (8.9) vanishes, that is to say where

$$KL = -\tfrac{1}{2}. \tag{8.17}$$

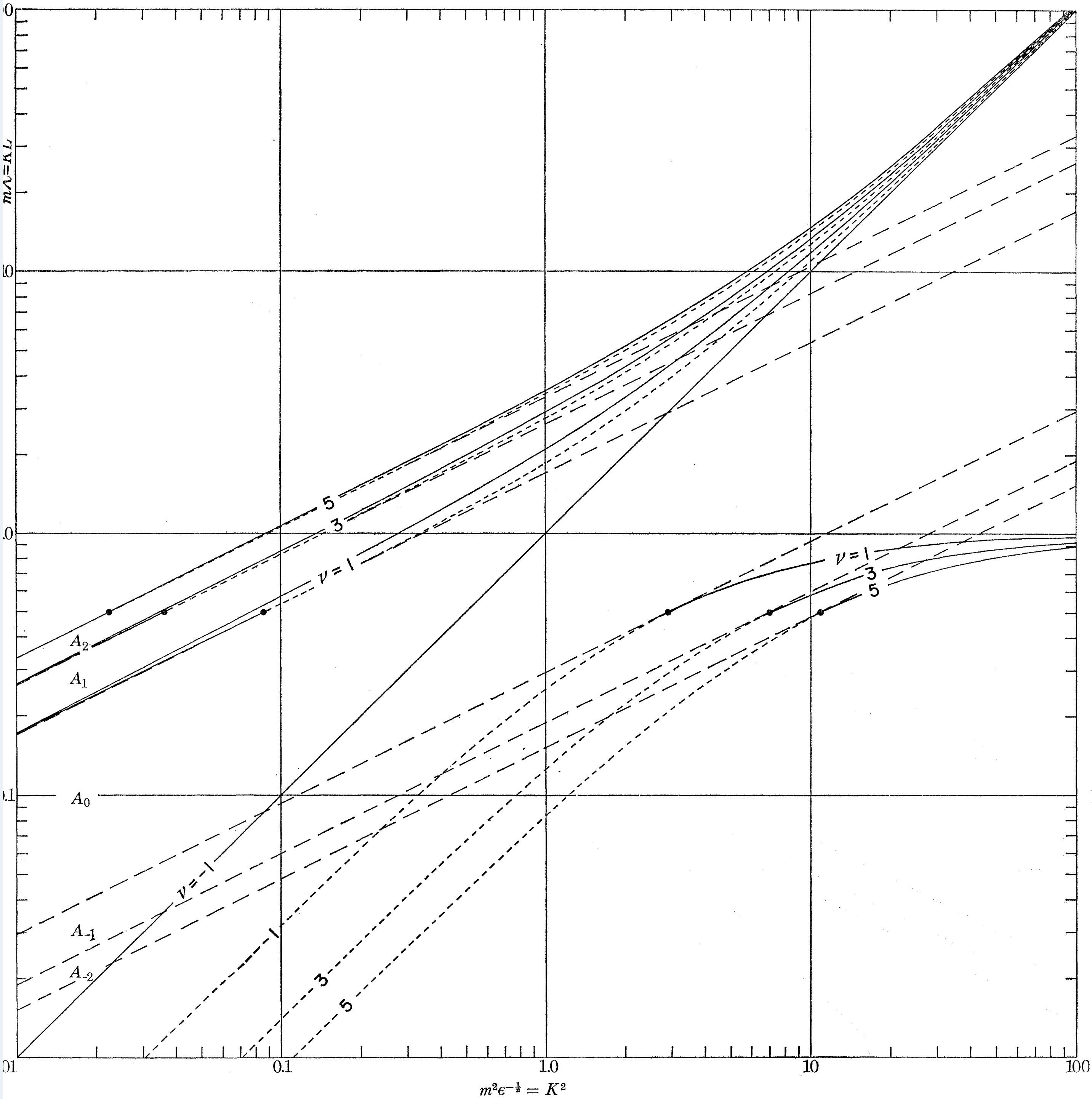


FIGURE 4. The relation between $m\lambda$ and $m^2\epsilon^{-\frac{1}{2}}$ for equatorial $\beta$-plane waves ($\epsilon \gg 1$) when $\Phi$ is symmetric about the equator.




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(The points of contact are marked in figure 4 by solid black circles.) Thus above the line $|m\lambda| = \frac{1}{2}$ all the westward modes of type 1 have westward group-velocity and all modes of type 2 have eastward group-velocity. Below the line $|m\lambda| = \frac{1}{2}$ all westward modes of type 1 have eastward group-velocity and all modes of type 2 have westward group-velocity.

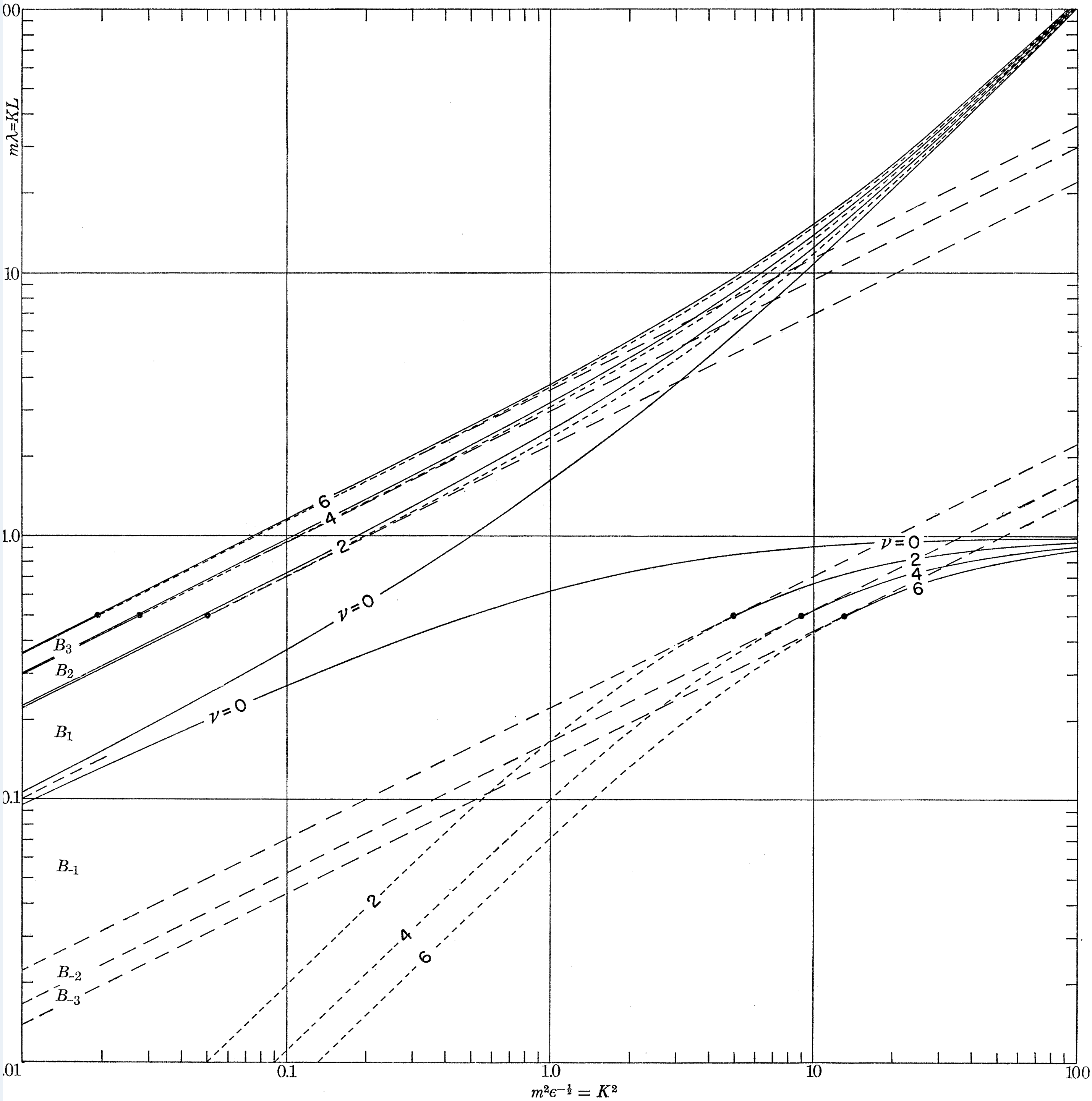


FIGURE 5. The relation between $m\lambda$ and $m^2\epsilon^{-\frac{1}{2}}$ for equatorial $\beta$-plane waves ($\epsilon \gg 1$) when $\Phi$ is antisymmetric about the equator.

The eastward modes of type 1, like the Kelvin wave, have only eastward group-velocity.

The antisymmetric curves, with $\nu$ even, are shown in figure 5. These are qualitatively similar to the curves of figure 4, except for the absence of the Kelvin wave. Instead we have the anti-Kelvin wave ($\nu = 0$), which can be seen to split into two branches, one east-going and the other west-going. Both have eastward group-velocities.

Taken together, the curves of figures 4 and 5 behave asymptotically (when $\epsilon \gg 1$) like the computed frequency curves of oscillations covering the whole sphere (Longuet-Higgins 1968*a*, figures 2 to 6). In that case the wavenumber $m$ (or $s$) is a non-negative integer determined by the condition that the solution be single-valued over the sphere.

The limitation on the curves of figures 4 and 5 for large values of $\epsilon^{-\frac{1}{2}}$ can be seen clearly by considering the behaviour as $m^2\epsilon^{-\frac{1}{2}} \to \infty$. In figures 4 and 5 we have for the type 1 curves, if $\nu$ is fixed,

$$|m\lambda| \sim m^2\epsilon^{-\frac{1}{2}}, \quad \lambda \sim \pm m\epsilon^{-\frac{1}{2}}, \tag{8.18}$$

whereas for class 1 waves (gravity waves) on a sphere

$$\lambda = \pm\sqrt{\{n(n+1)\}}\,\epsilon^{-\frac{1}{2}} \tag{8.19}$$

(Longuet-Higgins 1968, § 4). Equations (8.18) and (8.19) may be reconciled by noting that if both $m^2\epsilon^{-\frac{1}{2}} \ll 1$ and $\epsilon^{-\frac{1}{4}} \ll 1$ then we must have $m^2 \gg 1$. Since further

$$n = m + \nu, \tag{8.20}$$

where $\nu$ is fixed and of order 1, it follows that $\sqrt{\{n(n+1)\}} \sim m$, bringing (8.18) and (8.19) into agreement.

Likewise for the type 2 curves in figures 4 and 5 we have as $m^2\epsilon^{-\frac{1}{2}} \to \infty$

$$m\lambda \sim -1, \quad \lambda \sim -1/m, \tag{8.21}$$

compared with the limiting frequency for class 2 (planetary) waves:

$$\lambda = -m/n(n+1). \tag{8.22}$$

These are equivalent under the same conditions as before.

So for the asymptotic validity of the curves on the curves on the right of figures 4 and 5 we require $m \gg 1$. In general the curves are valid if the abscissa is small compared to $m^2$.

It can be seen that the tangents to the frequency curves, which are given by equations (8.13) and (8.14), divide the whole of figures 4 and 5 into zones, characterized by the presence or absence of certain types of wave motion. For example in figure 4, the central zone $A_0$, defined by

$$(\tfrac{3}{2} - \sqrt{2}) < \epsilon^{\frac{1}{2}}\lambda^2 < (\tfrac{3}{2} + \sqrt{2}), \tag{8.23}$$

contains only Kelvin waves. The zone above it, defined by

$$(\tfrac{3}{2} + \sqrt{2}) < \epsilon^{\frac{1}{2}}\lambda^2 < (\tfrac{7}{2} + \sqrt{2}), \tag{8.24}$$

contains only Kelvin waves and waves of type 1 with $\nu = 1$. In the zone above that we add the type 1 waves with $\nu = 2$, and so on. In general if we write

$$\left.\begin{aligned}(\nu + \tfrac{1}{2}) + \sqrt{\{\nu(\nu+1)\}} &= P_\nu^2,\\ (\nu + \tfrac{1}{2}) - \sqrt{\{\nu(\nu+1)\}} &= Q_\nu^2,\end{aligned}\right\} \tag{8.25}$$

then the zone $A_n$ defined by

$$P_{n-1} < \epsilon^{\frac{1}{4}}|\lambda| < P_n \tag{8.26}$$

contains the Kelvin wave and all waves of type 1 up to $\nu = n$; and the zone $A_{-n}$ defined by

$$Q_n < \epsilon^{\frac{1}{4}}|\lambda| < Q_{n-1} \tag{8.27}$$

contains the Kelvin wave and all waves of type 2 up to $n = \nu$.

Similar definitions may be made for the antisymmetric modes in figure 5.

It is worth noting, though it is hard to see in the diagrams, that each of the zones $A_n$, $B_n$ $(n > 0)$ is in fact split into two zones by the line

$$\epsilon^{\frac{1}{4}}|\lambda| = \sqrt{(2n+1)}. \tag{8.28}$$

Below this line only the west-going type 1 waves with $\nu = n$ can occur; above this line both east-going and west-going waves are possible.

## 9. The introduction of meridianal boundaries

The introduction of closed boundaries crossing the equator has the effect of determining the discrete sequence of wavenumbers $m$ for which the asymptotic approximations described in § 8 are applicable.

We note first that if meridianal boundaries exist it is not necessary for the east–west wavenumber $m$ (or equivalently $K$) to be real; we may contemplate expressions which increase or decrease exponentially on the far side of the boundary, but whose behaviour parallel to the boundary is sinusoidal. In fact Moore (1968) has shown that by a series of such terms it is possible to satisfy the condition $\boldsymbol{u}.\boldsymbol{n} = 0$ on a meridian, and that far from the equator such series can represent a Kelvin wave travelling along a meridian with the boundary to the right of the direction of propagation in the northern hemisphere, to the left in the southern hemisphere. Such solutions for a hemispherical basin were anticipated on physical grounds by Longuet-Higgins (1968*a*).

To fix the ideas, let us first consider oscillations in which $\Phi$ is symmetric about the equator. In figure 4, the central zone $A_0$ contains, as we saw, only Kelvin waves, whose energy is propagated eastwards along the equator (the group-velocity being positive). On meeting the eastern boundary the wave energy is presumably turned north or south and propagated away from the equator in the form of Kelvin waves. Presumably the energy passes close to the poles, and is then returned to the western end of the equator, to complete the circuit (see figure 30 of Longuet-Higgins 1968*a*).

If this interpretation is correct, then it should be possible to calculate approximately the frequencies of the corresponding normal modes, as follows. The phase velocity of the Kelvin waves, both along the equator and along the meridianal boundary, is equal to $\sqrt{(gh)}$ independently of $f$. The total path length is equal to $\pi$ along the equator (the radius of the Earth being taken as unity) plus $\pi$ along the meridianal boundary—a total of $2\pi$. We neglect small differences in path-length, of order $\epsilon^{-\frac{1}{4}}$, at each end of the equatorial path. There may, however, be a finite phase-delay $(\pi+\delta)$ in the pressure as the wave turns the corner from equator to meridian and vice versa. Hence if $m$ denotes the wavenumber at the equator, the total phase change in one circuit equals

$$2m\pi + 2\delta = 2s\pi, \tag{9.1}$$

say, where $s$ must be a positive integer. Therefore we have

$$m = s - \delta/\pi. \tag{9.2}$$

The non-dimensional frequency $\lambda$ is given by

$$\lambda = \sigma/2\Omega = m\sqrt{(gh)}/2\Omega = \epsilon^{-\frac{1}{2}}(s-\delta/\pi). \tag{9.3}$$

Conversely,

$$\delta/\pi = s - \epsilon^{\frac{1}{2}}\lambda. \tag{9.4}$$

Now the phase-change $\delta$ has been calculated independently by Moore (1968) for a *rectangular* basin on a $\beta$-plane. Moore uses a series expansion of the solutions of (8.3) in order to satisfy the boundary condition $u = 0$ on the meridianal boundaries. Neglecting quantities of order $\epsilon^{-\frac{1}{4}}$ Moore finds

$$\delta = \arcsin\left[\left(\frac{1}{2L}-L\right)\Big/\sqrt{2}\right]+C\left[\frac{1}{2}\left(\frac{1}{2L}-L\right),\frac{1}{2}\right]-C\left[\frac{1}{2}\left(\frac{1}{2L}+L\right),0\right], \tag{9.5}$$

where
$$C(a,b) \equiv \lim_{r\to\infty}\left[\sum_{q=1}^{\infty}\arcsin\left(\frac{a}{b+q}\right)-2a(b+q)^{\frac{1}{2}}\right]. \tag{9.6}$$

Figure 6 shows the comparison between Moore's expression† for $\delta/\pi$ and the values derived from equation (9.4), where $\lambda$ denotes the frequency of waves in the zone $A_0$ of figure 1 (and computed

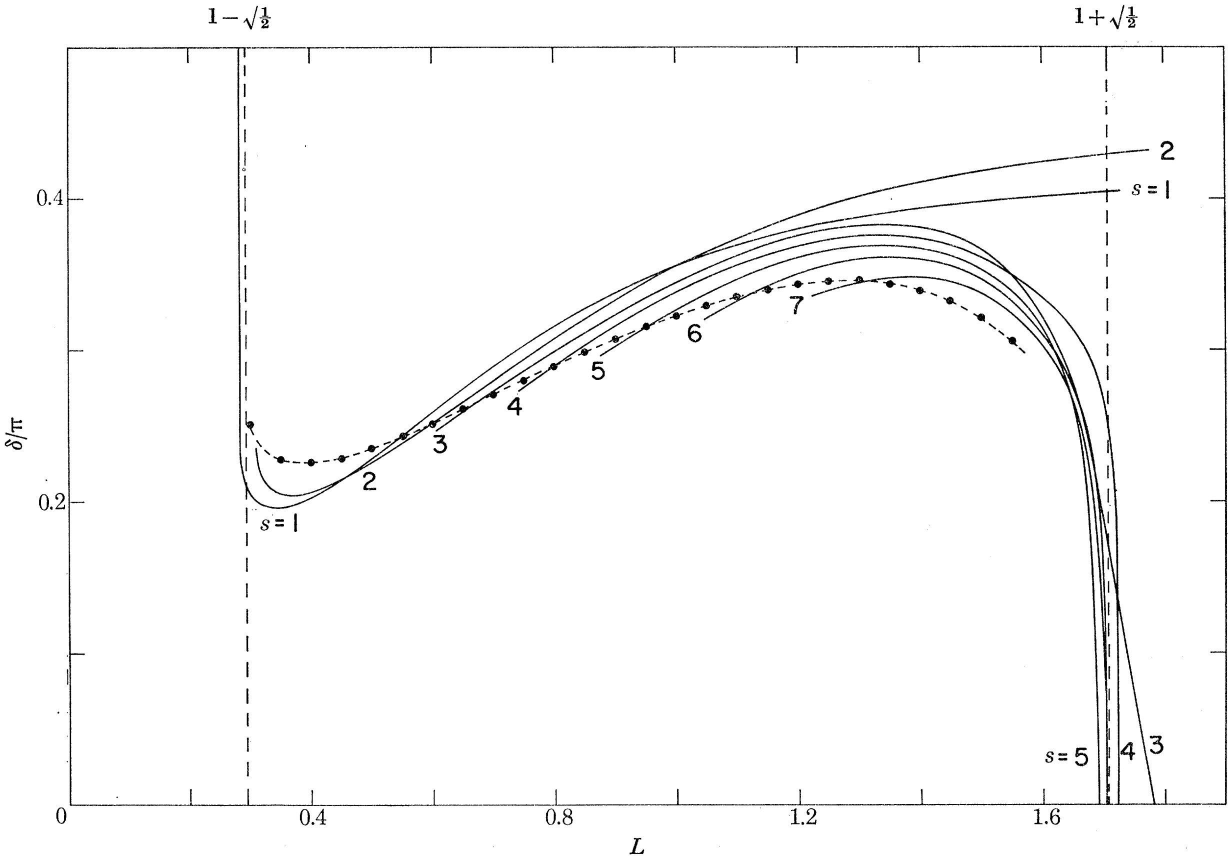


FIGURE 6. The calculated phase difference $\delta/\pi$ as given by equation (10.4) for the symmetric modes in the central zone $A_0$ of figure 1 (Kelvin waves). The dashed curve shows the value of $\delta/\pi$ derived from the equatorial $\beta$-plane approximation (equation (10.5)).

in §§ 2 to 6 of the present paper) and $s$ denotes the nearest integer. As can be seen from figure 6, the qualitative agreement is good, especially for the larger values of $s$. This confirms that our interpretation of the curves in the zone $A_0$ as Kelvin waves is correct.

A similar analysis can be carried out for the anti-Kelvin waves. In such waves the northwards component of velocity near the equator has the form

$$v = V\exp\{-\tfrac{1}{2}\eta^2\}\exp\{\mathrm{i}(K\xi-L\tau)\}, \tag{9.7}$$

where $V$ is a constant and $K$ is given by

$$K = L-1/L \tag{9.8}$$

† The numerical calculations given by Moore in the original version of his thesis have been revised. I am indebted to Dr Moore for supplying me with the corrected values.

from (8.10). Hence the phase velocity along the equator is

$$c = \sigma/m = (L/K)\surd(gh) = \frac{L^2\surd(gh)}{L^2-1}, \tag{9.9}$$

and is positive or negative according as $L^2 \gtrless 1$. The group-velocity, however, is always positive, so that the energy always flows eastwards along the equator and must be returned via the poles by a Kelvin wave along the meridianal boundary. Now if $\lambda = \epsilon^{-\frac{1}{4}}L$ denotes the non-dimensional frequency, the phase-change along the equator (at any instant of time) is given by

$$m\pi = \epsilon^{\frac{1}{4}}K\pi = (L-1/L)\,\pi\epsilon^{\frac{1}{4}}. \tag{9.10}$$

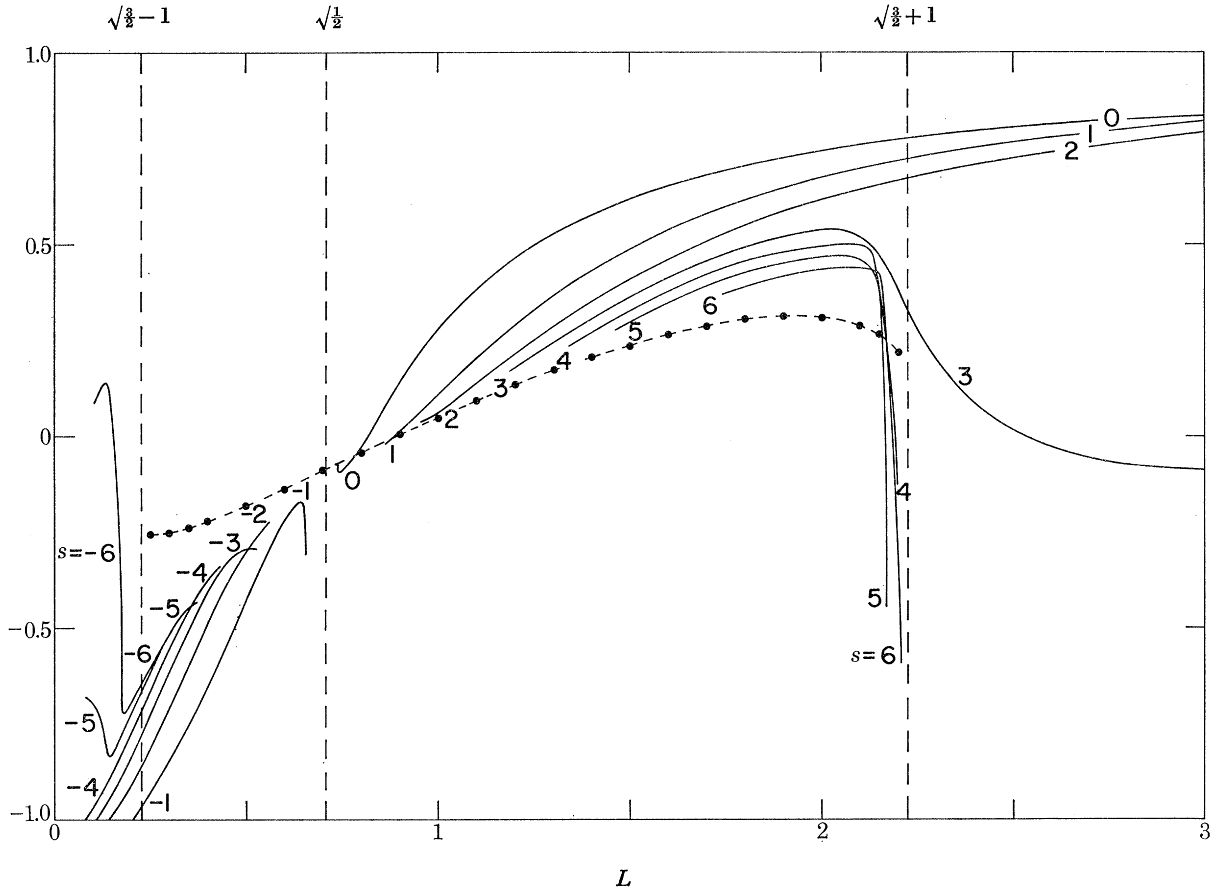


FIGURE 7. The calculated phase difference $\delta'/\pi$ as given by equation (10.14) for the antisymmetric modes in the central zones $B_1$ and $B_{-1}$ of figure 2 (anti-Kelvin waves). The dashed curve shows Moore's calculated value of $\delta'/\pi$ derived from the equatorial $\beta$-plane approximation.

The phase-change in $v$ along the meridianal boundary is

$$(L\pi\epsilon^{\frac{1}{4}}+\pi), \tag{9.11}$$

the second term $\pi$ arising because on the western boundary the southwards-going Kelvin wave has a phase velocity in the opposite direction to the northwards wave on the eastern boundary. The phase of $v$ is reversed on passage through the pole. Then if we denote by $\delta'$ the phase delay resulting from turning a corner at one end of the equatorial path we have altogether

$$(L-1/L)\,\pi\epsilon^{\frac{1}{4}} + (L\pi\epsilon^{\frac{1}{4}}+\pi) + 2\delta' = 2s'\pi, \tag{9.12}$$

where $s'$ is an integer. Hence

$$\delta'/\pi = (s'-\tfrac{1}{2}) - \epsilon^{\frac{1}{4}}(L-1/2L) \tag{9.13}$$

$$= (s'-\tfrac{1}{2}) - (\epsilon^{\frac{1}{2}}\lambda - 1/2\lambda). \tag{9.14}$$

The values of $\delta'/\pi$ computed from those curves which lie in the zones $B_1$ and $B_{-1}$ of figure 5 have been plotted in figure 8 (continuous curves) along with the phase-shifts as computed by Dr D. W. Moore (personal communication) from the series expansions of the velocity field. Again the agreement is satisfactory, and improves as $s$ is increased. By including negative values of $s'$, both east-going and west-going anti-Kelvin waves may be conveniently shown on the same diagram.

In a similar way one might hope to find recognizable normal modes in other zones of figures 1 and 2, wherever there was an equatorial wave with an eastward *group*-velocity (the phase velocity seems immaterial) so that the eastward flux of energy along the equator might be completed by Kelvin waves along the meridianal boundaries. In zone $A_2$ for example, in figure 4, the east-going waves of type 1 have this property, as do those west-going waves of type 2 such that $|m\lambda| < \frac{1}{2}$. Similarly, in zone $A_{-2}$ the west-going waves of type 2 whose frequencies are such that $\frac{1}{2} < |m\lambda| < 1$ also have the required property of eastward group-velocity.

However, in all but the lowest zones $A_0$ and $B_{\pm 1}$ an important difficulty arises, namely, that at each point more than one type of asymptotic solution is theoretically possible. Each zone, therefore, except the lowest, is to be crossed by more than one system of curves. A confused pattern of solutions then results which neither the eye nor the computer is able to resolve satisfactorily. Moreover the asymptotic solutions perturb one another in a way that will be discussed in the following section.

## 10. Continuity of the eigenvalues

We have considered so far only the asymptotic behaviour of the eigenfrequencies as $\epsilon \to 0$ or $\infty$. The way in which the asymptotic values are connected over the middle range of frequencies can most easily be seen in figures 1 and 2. In figure 1, for example, it is clear that those class 1 waves on the right of the diagram which have $(n-m) = 0$ tend to be connected with the Kelvin waves on the left, which have $\nu = -1$. Those class 1 waves which have $(n-m) = 2$ tend to be connected with the type 1 waves having $\nu = +1$; and so on.

Similarly, in figure 2, the class 1 waves with $(n-m) = 1$ tend to be connected with the east-going anti-Kelvin waves $(\nu = 0)$ and the class 1 waves with $(n-m) = 3$ tend to be connected to type 1 waves with $\nu = 2$, etc.

Similar connexions may be traced for the class 2 modes (the planetary waves). In general those class 2 modes having $(n-m) = p$ tend to be connected to type 2 waves with index $\nu = p-1$.

All those connexions among the modes are found also for waves on the unbounded sphere (Longuet-Higgins 1968*a*, figures 2 to 6).

However, in the present situation with boundaries, there are clearly many instances where, if the foregoing connexions among the modes were to be continuous, the corresponding frequency curves would have to intersect. For example in figure 2, some class 2 waves with $\nu = 0$ (on the right of the diagram) have a *lower* frequency than some class 2 waves with $\nu = 2$. If these were to be connected with the appropriate type 2 waves on the left of the diagram there would have to be some intersections at intermediate values of $\epsilon$.

A close inspection of the computed frequency curves shows that in fact two curves do not intersect, though they may come near to doing so. On the contrary, near the expected point of intersection the curves behave like the two branches of hyperbola, as in figure 8. The two modes exchange roles, as it were.

This behaviour may be understood by taking a system of normal coordinates for the modes in which each of the modes is represented by a single coordinate $q_i$ $(i = 1, 2)$ dependent only on the time $t$. In this system of coordinates the equations of motion for the $q_i$ in the vicinity of an intersection will take the form

$$\left.\begin{aligned} d^2q_1/dt^2 + Aq_1 + \alpha q_2 &= 0, \\ d^2q_2/dt^2 + \beta q_1 + Bq_2 &= 0, \end{aligned}\right\} \tag{10.1}$$

where $A$ and $B$ are functions of $\epsilon$ both of which are equal to $\sigma_0^2$ when $\epsilon = \epsilon_0$; and where $\alpha$ and $\beta$ are relatively small coupling constants. Thus in the neighbourhood of $\epsilon_0$ each of the modes tends to oscillate with a frequency $\sigma$ close to $\sigma_0$. Assuming that the frequency of the combined motion is equal to $\sigma$ we have from (10.1)

$$\left.\begin{aligned} (\sigma^2 - A)\, q_1 &= \alpha q_2, \\ (\sigma^2 - B)\, q_2 &= \beta q_1. \end{aligned}\right\} \tag{10.2}$$

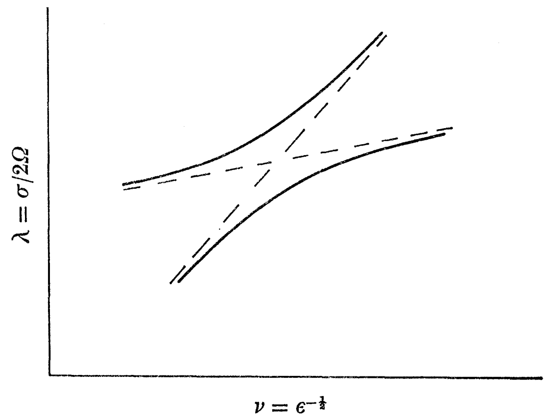


FIGURE 8. Enlarged sketch of the form of the frequency curves in the neighbourhood of a near-intersection.

The condition that these equations have a consistent solution is that

$$(\sigma^2 - A)(\sigma^2 - B) = \alpha\beta. \tag{10.3}$$

Now in accordance with our previous assumptions write

$$\sigma = \sigma_0 + \Delta\sigma, \qquad \epsilon = \epsilon_0 + \Delta\epsilon, \tag{10.4}$$

so that

$$A \doteqdot \sigma_0 + \frac{\partial A}{\partial \epsilon}\Delta\epsilon, \qquad B \doteqdot B_0 + \frac{\partial B}{\partial \epsilon}\Delta\epsilon.$$

Then to order $(\Delta\epsilon)^2$ equation (11.3) becomes

$$\left(2\sigma_0 \Delta\sigma - \frac{\partial A}{\partial \epsilon}\Delta\epsilon\right)\left(2\sigma_0 \Delta\sigma - \frac{\partial B}{\partial \epsilon}\Delta\epsilon\right) = \alpha\beta, \tag{10.5}$$

which represents a hyperbola with asymptotes

$$2\sigma_0 \Delta\sigma = \frac{\partial A}{\partial \epsilon}\Delta\epsilon, \qquad 2\sigma_0 \Delta\sigma = \frac{\partial B}{\partial \epsilon}\Delta\epsilon, \tag{10.6}$$

To the order $(\Delta\epsilon)^2$ the asymptotes may be written

$$(\sigma^2 - A)(\sigma^2 - B) = 0. \tag{10.7}$$

In other words, the asymptotes of the hyperbola represent the form that the two frequency curves would take in the absence of any coupling $(\alpha\beta = 0)$. Because a hyperbola consists of two branches, running from one asymptote to the other, we see that in the presence of coupling the two frequency curves simply exchange roles.

It may be mentioned that a similar exchange of frequencies is found empirically in layered media with two or more dominant propagation channels (Press & Harkrider 1962; Pfeffer & Zarichny 1963) and probably in many other physical situations as well.

It can also be shown that in the neighbourhood of such an exchange point the actual normal modes $p_j$, $p_k$ consist not of $q_1$ and $q_2$ separately, but two independent linear combinations of $q_1$ and $q_2$. These two combinations have slightly different frequencies $\sigma_j$ and $\sigma_k$, say. Under general initial conditions, both of the modes $p_j$ and $p_k$, with their corresponding frequencies, will be excited, and then beating between the frequencies $\sigma_j$ and $\sigma_k$ will give the appearance of a slow exchange of energy between $q_1$ and $q_2$.

In this way, two or more asymptotically distinct types of motion may exchange energy between each other, the necessary condition being only that they have almost the same frequency. If the waves are undamped, as in the present theory, then for most initial conditions we may expect that the mutual exchange of energy will result in an equipartition of energy between the modes, on average. But any viscous damping, which will affect some modes more than others, will result in the virtual absence of the more highly damped modes.

## 11. The negative values of $\epsilon$

As mentioned in §5, the computations yielded also some eigenvalues $\lambda$ corresponding to negative values of $\epsilon$. Similar solutions appeared in the problem of oscillations on a complete globe (Longuet-Higgins 1968*a*), and their interpretation was discussed in §§10 to 13 of that paper. Generally speaking, such solutions would not be realizable as *free* oscillations in a stable ocean; however, they are essentially useful in determining the response of the ocean to external forces of a given frequency, such as gravitational tide-raising forces.

The computed values of $\lambda$ are shown in figure 9 for $\Phi$ symmetric, $\Psi$ antisymmetric, and in figure 10 for $\Phi$ antisymmetric and $\Psi$ symmetric. The curves in figures 9 and 10 bear a general resemblance to the corresponding curves for the westgoing, negative-depth modes on the unbounded sphere (Longuet-Higgins 1968*a*; figures 17*b* to 21*b*). For example, as $\epsilon \to -0$ they tend to the frequencies of the non-divergent oscillations. These have been tabulated already in table 2. As $\epsilon \to -\infty$, the eigenfrequencies $\lambda$ tend apparently to $\pm 1$, so that the motions have the inertial frequency.

The asymptotic form of these solutions as $\epsilon \to \infty$ can be found from the analysis given in §11 of the paper just referred to. If first we ignore the presence of the meridianal boundary and look for solutions proportional to $\exp\{i(s\phi - \sigma t)\}$, where $s$ is a real wavenumber, then we find that as $\epsilon \to -\infty$ the motion must tend to be concentrated towards the poles. Hence we can study the solution analytically by adopting a polar $\beta$-plane approximation that is to say by writing

$$f = 2\Omega(1 - \tfrac{1}{2}\theta^2), \quad \beta = \Omega\theta/R, \tag{11.1}$$

where $\theta$ is the angular distance from the pole (as in Leblond 1964). We introduce the scaled radial distance

$$\omega = (-\beta)^{\frac{1}{4}}\theta. \tag{11.2}$$

There are two cases of interest; for westgoing waves ($s \geq 1$, $\lambda < 0$) we find that $\lambda \doteqdot -1$. Hence writing

$$\lambda = -1 + \frac{Q}{(-\epsilon)^{\frac{1}{2}}} + O\left(\frac{1}{\epsilon}\right) \tag{11.3}$$

we obtain as an equation for $v^* = v \sin\theta$ the following:

$$\left(\frac{\partial^2}{\partial\omega^2} - \frac{1}{\omega}\frac{\partial}{\partial\omega} - \frac{s^2-2s}{\omega} + 2Q\right) v^* = 0. \tag{11.4}$$

This has solutions of the form

$$v^* \propto \exp\{-\tfrac{1}{2}\omega^2\}\,\omega^s L_\nu^{(s-1)}(\omega^2)\exp\{\mathrm{i}(s\phi - \sigma t)\}, \tag{11.5}$$

where $L_\nu^a(z)$ denotes the generalized Laguerre polynomial

$$L_\nu^a(z) \equiv \sum_{m=0}^{\infty} \binom{\nu+a}{\nu-m} \frac{(-z)^m}{m!}, \tag{11.6}$$

and $\nu$ is a non-negative integer related to $Q$ by

$$Q = 2s + 2\nu - 1. \tag{11.7}$$

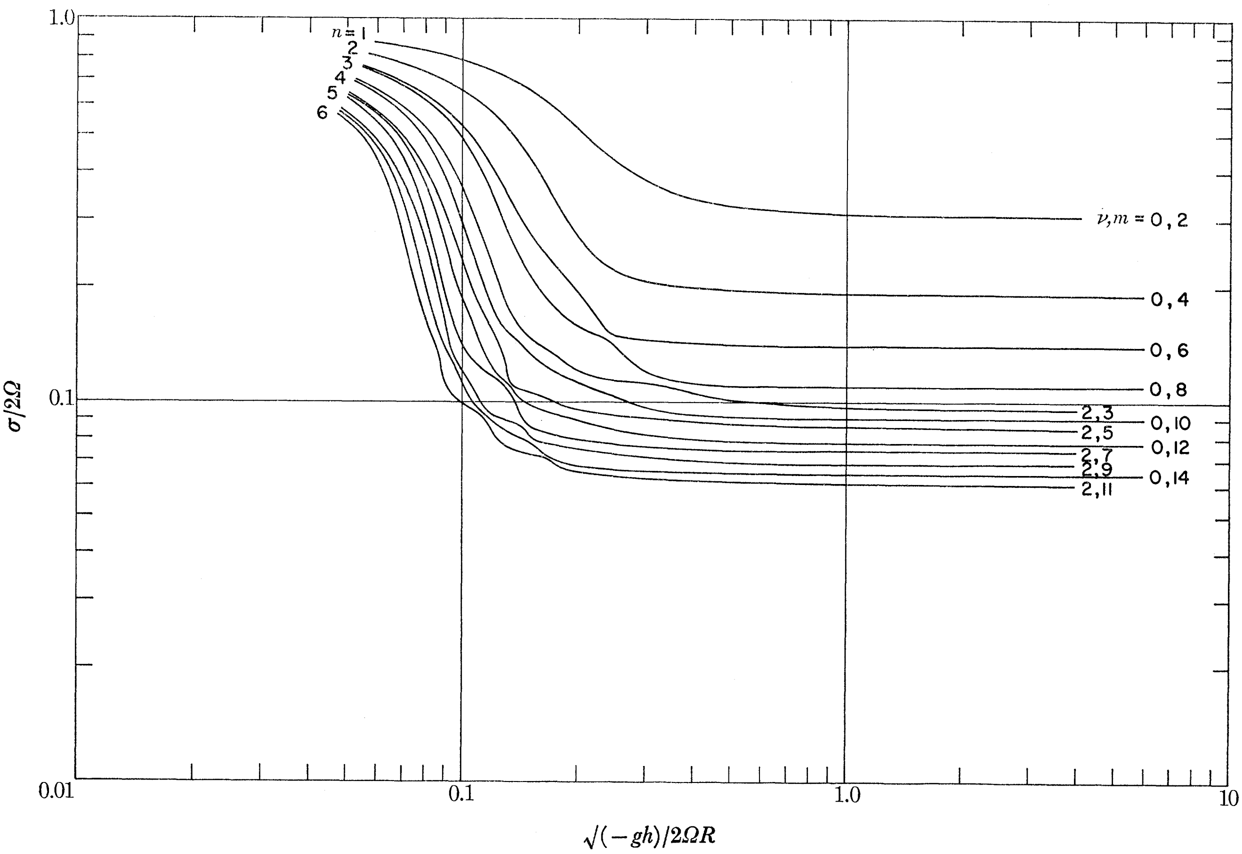


FIGURE 9. The computed eigenfrequencies of the 'negative-depth' modes when $\Phi$ is symmetric about the equator.

The velocity $u^* = u \sin\theta$ is related to $v^*$ by

$$u^* = -\mathrm{i}v^* \tag{11.8}$$

approximately so that the motion is roughly in inertial circles.

For eastgoing waves ($s \geqslant 1$, $\lambda > 0$) we find two types of limiting solutions as $\epsilon \to -\infty$. In the first of these $\lambda \doteqdot 1$, and hence

$$v^* \propto \exp\{-\tfrac{1}{2}\omega^2\}\,\omega^{s+2} L_\nu^{(s+1)}(\omega^2)\exp\{\mathrm{i}(s\phi - \sigma t)\} \tag{11.9}$$

with
$$\lambda = 1 - \frac{2s+2\nu+3}{(-\epsilon)^{\frac{1}{2}}} + O\left(\frac{1}{\epsilon}\right). \tag{11.10}$$

In this type of motion
$$u^* \doteqdot iv^* \tag{11.11}$$
so that again the particles move roughly in inertial circles.

The second type of eastward motion, in which $\lambda \doteqdot s/(-\epsilon)$, will not concern us here.

Now to satisfy the boundary condition $u^* = 0$ when $\phi = 0, \pi$ we write $(s+1)$ for $s$ in (11.5), $(s-1)$ for $s$ in (11.9) and change i to $(-i)$ in (11.9) and (11.11), (so that $u^* = -iv^*$ for both solutions). Subtracting the solutions we obtain
$$v^* \propto \exp\{-\tfrac{1}{2}\omega^2\}\,\omega^{s+1}L_\nu^s(\omega^2)\exp\{i\phi\}\sin s\phi\exp\{-2i\Omega t\}, \tag{11.12}$$
provided
$$s \geqslant 2. \tag{11.13}$$

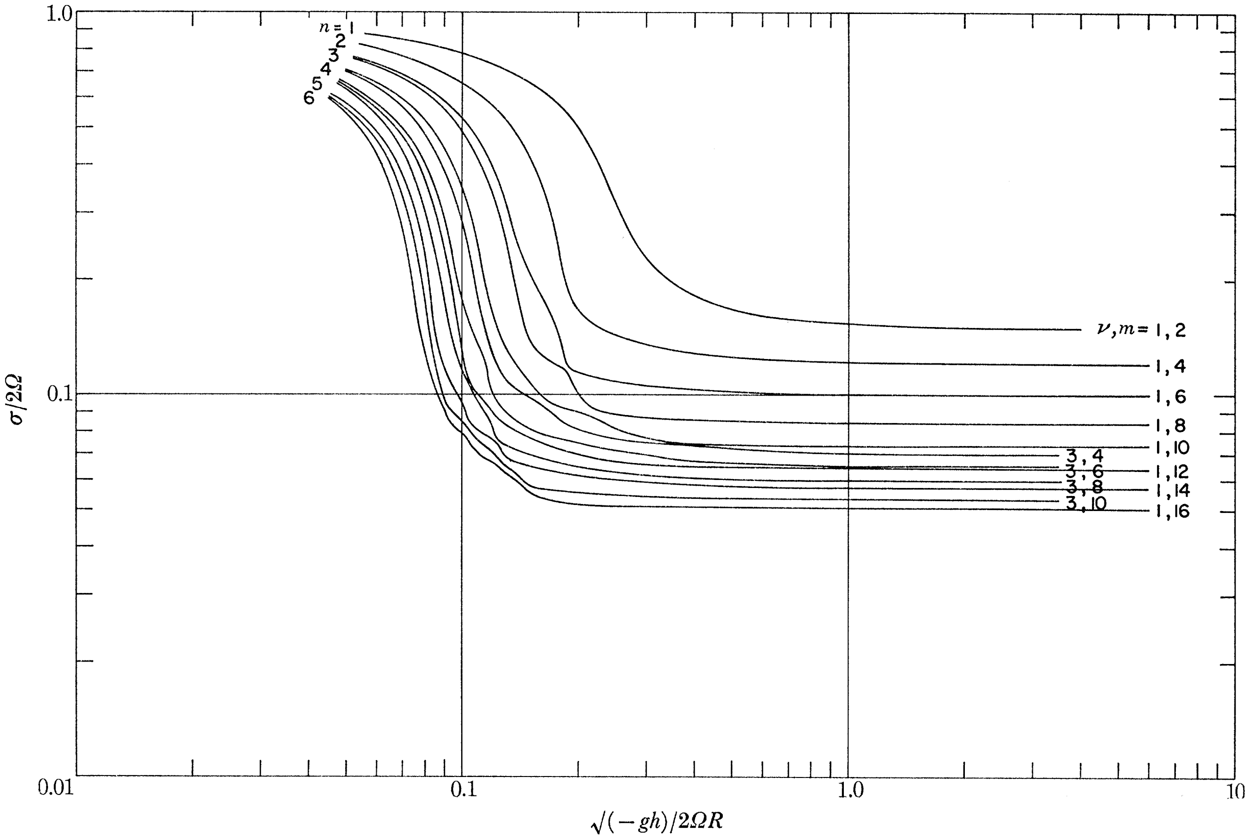


FIGURE 10. The computed eigenfrequencies of the 'negative-depth' modes when $\Phi$ is antisymmetric about the equator.

Since $u^* = -iv^*$ this satisfies the boundary condition at $\phi = 0, \pi$. The frequency is given by
$$\lambda = 1 - \frac{2s+2\nu+1}{(-\epsilon)^{\frac{1}{2}}} + O\left(\frac{1}{\epsilon}\right). \tag{11.14}$$

The solution (11.12) represents a carrier wave of the form
$$\exp\{i(\phi - 2\Omega t)\}, \tag{11.15}$$
progressing eastwards, modulated by an amplitude function of the form
$$\exp\{-\tfrac{1}{2}\omega^2\}\,\omega^{s+1}L_\nu^s(\omega^2)\sin s\phi. \tag{11.16}$$

The asymptotic form (11.14) of the frequency agrees well with the computed values of $\lambda$ in figures 9 and 10, when $-\epsilon \gg 1$. The corresponding values of $n = \nu + s$ have been noted against each curve.

## 12. Conclusions

The present calculation has extended the previous computation of the non-divergent modes in a hemispherical basin so as to take full account of the horizontal divergence of the motion. Figures 1 and 2 show, for example, that when the parameter $\epsilon$ is equal to 20 (typical for barotropic waves) and so $\sqrt{(gh)}/2\Omega R = 0.22$ the frequency of the lowest class 2 mode (figure 1) is less than the non-divergent limit by about 30 %. Since the vertical scale in figures 1 and 2 are logarithmic, it can easily be seen that the other modes are altered by a smaller amount.

The calculations also show the extent to which the frequencies of the normal modes approach simple asymptotic forms, both for large and small values of $\epsilon$. Thus by an appropriate use of the asymptotic expressions, one may be able to estimate quite accurately the normal mode frequencies (barotropic and baroclinic) in ocean basins of other shapes and dimensions (but still of uniform depth).

Although it would be easy in principle to extend the present method of calculation so as to obtain the normal modes in basins bounded by meridians which were separated by any angle other than 180°, for example 60° or 120° this may not be necessary; for we have shown that in some circumstances quite close approximate values can be supplied by reference to the asymptotic forms mentioned earlier.

For example, in an ocean bounded by meridians which are separated by an angle of 120°, the period of the equatorial Kelvin wave can be estimated from the fact that (1) the total path-length is equal to $(\frac{5}{3}\pi R)$; (2) the speed of propagation equals $\sqrt{(gh)}$; and (3) the phase change $\delta$ at each end of the equatorial section of the path is given approximately by the broken line in figure 7.

The influence of variable depth $h$ can also be studied, in theory, by the powerful method of Proudman (1916) which has been used in the present study. Nevertheless, the results will probably be understood best in conjunction with simpler models which show the effects of local topography in trapping or scattering wave energy (see, for example, Rhines 1967, 1969; Longuet-Higgins 1968*b*, *c*; Buchwald 1969).

For simplicity we have assumed a linear, non-viscous model. But in practice scattering by isolated bottom features, combined with damping may result in important losses of wave energy.

## Appendix. Evaluation of the gyroscopic coefficients

We wish to evaluate the coefficients defined by equations (2.9) when $\Phi_r$ and $\Psi_r$ are given by (3.1) and (3.8) respectively. We take $\binom{m}{n}$ corresponding to the suffix $r$, and $\binom{m'}{n'}$ corresponding to $s$.

Consider first the integration with respect to $\phi$. For all integer values of $m$, $m' \geqslant 0$ we have

$$\int_0^\pi m \sin m\phi \cos m\phi \, d\phi = \begin{cases} \dfrac{2m^2}{m^2 - m'^2}, & (m+m') \text{ odd}, \\ 0, & (m+m') \text{ even}. \end{cases} \tag{A 1}$$

Hence we need confine attention only to the case when $(m+m')$ is odd. By straightforward substitution and integration with respect to $\phi$ we have then

$$\left.\begin{aligned}\frac{\beta_{r,s}}{\alpha_r\alpha_s} &= \frac{2}{m^2-m'^2}\int_{-1}^{1}\left(m^2P_n^m\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}+m'^2\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}P_{n'}^{m'}\right)\mu\,\mathrm{d}\mu,\\ \frac{\beta_{-r,s}}{\alpha_r\alpha_s} &= \frac{-2m}{m^2-m'^2}\int_{-1}^{1}\left[(1-\mu^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}+\frac{m'^2}{1-\mu^2}P_n^mP_{n'}^{m'}\right]\mu\,\mathrm{d}\mu,\\ \frac{\beta_{-r,-s}}{\alpha_r\alpha_s} &= \frac{2mm'}{m^2-m'^2}\int_{-1}^{1}\left(\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}P_{n'}^{m'}+P_n^m\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}\right)\mu\,\mathrm{d}\mu,\end{aligned}\right\} \tag{A 2}$$

with $\beta_{r,-s} = -\beta_{-s,r}$. We shall express each of these quantities in terms of integrals of the form

$$I\begin{pmatrix}m & m'\\ n & n'\end{pmatrix} = \int_{-1}^{1}P_n^mP_{n'}^{m'}\,\mathrm{d}\mu. \tag{A 3}$$

We use the following auxiliary results. First, since $(m+m')$ is odd, $m$ and $m'$ cannot both be zero and so by the definition (3.4)

$$P_n^mP_{n'}^{m'} = 0 \quad \text{when} \quad \mu = \pm 1. \tag{A 4}$$

Secondly, for all $m \geqslant 0$, differentiation of (3.4) gives

$$\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu} = \frac{1}{(1-\mu^2)^{\frac{1}{2}}}P_n^{m+1}-\frac{m\mu}{1-\mu^2}P_n^m \tag{A 5}$$

and on substituting in the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\left[(1-\mu^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]P_n^m = 0 \tag{A 6}$$

we find
$$P_n^{m+2}-\frac{2(m+1)\mu}{(1-\mu^2)^{\frac{1}{2}}}P_n^{m+1}+[n(n+1)-m(m+1)]P_n^m = 0. \tag{A 7}$$

Then eliminating $P_n^{m+1}$ between (A 5) and (A 7) we obtain

$$\mu\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu} = \frac{1}{2(m+1)}P_n^{m+2}+\left[\frac{n(n+1)+m(m+1)}{2(m+1)}-\frac{m}{1-\mu^2}\right]P_n^m. \tag{A 8}$$

Thirdly, if the differential equation (A 6) be multiplied by $P_{n'}^{m'}$ and integrated over $(-1 < \mu < 1)$ we have
$$\int_{-1}^{1}\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]P_n^mP_{n'}^{m'}\,\mathrm{d}\mu = \int_{-1}^{1}(1-\mu^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu}\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}\,\mathrm{d}\mu, \tag{A 9}$$

the right-hand side having been integrated by parts. Subtracting from (A 9) the equation got by interchanging $\begin{pmatrix}m\\ n\end{pmatrix}$ with $\begin{pmatrix}m'\\ n'\end{pmatrix}$ we obtain

$$\int_{-1}^{1}\left[\{n(n+1)-n'(n'+1)\}-\frac{m^2-m'^2}{1-\mu^2}\right]P_n^mP_{n'}^{m'}\,\mathrm{d}\mu = 0, \tag{A 10}$$

and so
$$\int_{-1}^{1}P_n^mP_{n'}^{m'}\frac{\mathrm{d}\mu}{1-\mu^2} = \frac{n(n+1)-n'(n'+1)}{m^2-m'^2}I\begin{pmatrix}m & m'\\ n & n'\end{pmatrix}. \tag{A 11}$$

Now to apply these results to the evaluation of the integrals (A 2), let us write the first of these in the form
$$\frac{\beta_{r,s}}{\alpha_r\alpha_s} = 2\int_{-1}^{1}\left[P_n^m\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}+\frac{m'^2}{m^2-m'^2}\frac{\mathrm{d}}{\mathrm{d}\mu}(P_n^mP_{n'}^{m'})\right]\mu\,\mathrm{d}\mu. \tag{A 12}$$

The second group of terms in the integrand can be integrated by parts to give

$$\frac{2m'^2}{m^2-m'^2}\left\{[P_n^m P_{n'}^{m'}\mu]_{-1}^{1}-\int_{-1}^{1} P_n^m P_{n'}^{m'}\,\mathrm{d}\mu\right\} = -\frac{2m'^2}{m^2-m'^2} I\begin{pmatrix} m & m' \\ n & n' \end{pmatrix} \tag{A 13}$$

by (A 3) and (A 4). The first group of terms in (A 12) can be transformed by means of (A 8) to give

$$2\int_{-1}^{1} P_n^m \left[\frac{1}{2(m'+1)} P_{n'}^{m'+2} + \frac{n'(n'+1)+m'(m'+1)}{2(m'+1)} P_{n'}^{m'}\right]\mathrm{d}\mu - 2m'\int_{-1}^{1} P_n^m P_{n'}^{m'} \frac{\mathrm{d}\mu}{1-\mu^2}, \tag{A 14}$$

and when use is made of (A 11) we find altogether equation (3.14).

To deal with the second of equations (A 2) we note that the integral may be written

$$\int_{-1}^{1} (1-\mu^2)\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}\mu\,\mathrm{d}P_n^m + \int_{-1}^{1}\frac{m'^2}{1-\mu^2} P_n^m P_{n'}^{m'}\mu\,\mathrm{d}\mu, \tag{A 15}$$

which on integration by parts becomes

$$-\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\mu}\left[\mu(1-\mu^2)\frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}\right] P_n^m\,\mathrm{d}\mu + \int_{-1}^{1}\frac{m'^2}{1-\mu^2} P_n^m P_{n'}^{m'}\mu\,\mathrm{d}\mu. \tag{A 16}$$

Then on using the differential equation (A 6) with $\begin{pmatrix} m \\ n \end{pmatrix}$ replaced by $\begin{pmatrix} m' \\ n' \end{pmatrix}$ this becomes

$$n'(n'+1)\int_{-1}^{1} P_n^m P_{n'}^{m'}\mu\,\mathrm{d}\mu - \int_{-1}^{1}(1-\mu^2) P_n^m \frac{\mathrm{d}P_{n'}^{m'}}{\mathrm{d}\mu}\mathrm{d}\mu \tag{A 17}$$

and, on using the known relations,

$$\left.\begin{aligned} \mu P_n^m &= \frac{n+m}{2n+1} P_{n-1}^m + \frac{n-m+1}{2n+1} P_{n+1}^m, \\ (1-\mu^2)\frac{\mathrm{d}P_n^m}{\mathrm{d}\mu} &= \frac{(n+1)(n+m)}{2n+1} P_{n-1}^m - \frac{n(n-m+1)}{2n+1} P_{n+1}^m \end{aligned}\right\} \tag{A 18}$$

$\left(\text{but with} \begin{pmatrix} m \\ n \end{pmatrix} \text{replaced by} \begin{pmatrix} m' \\ n' \end{pmatrix}\right)$ we can express (A 17) in terms of $I\begin{pmatrix} m & m' \\ n & n'-1 \end{pmatrix}$ and $I\begin{pmatrix} m & m' \\ n & n'+1 \end{pmatrix}$. This leads to equation (3.15).

In the third and last of equations (A 2) we note that the integral may be written simply

$$\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}\mu}(P_n^m P_{n'}^{m'})\mu\,\mathrm{d}\mu = [\mu P_n^m P_{n'}^{m'}]_{-1}^{1} - I\begin{pmatrix} m & m' \\ n & n' \end{pmatrix}. \tag{A 19}$$

The first term on the right vanishes by (A 4) and we obtain immediately equation (3.17).

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